Probability behind Spatial Stochastic Frontier model

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Outline

- Classical stochastic frontier model
- A case of truncated normal inefficiency
- Spatial stochastic frontier model
- Moment generating and characteristic functions
- Closed/Unified skew-normal distribution

Economic background of stochastic frontier model

We consider a company, which uses *K* inputs, indexed *k* = 1, 2, …, *K*, to produce *M* outputs, indexed *m* = 1, 2, …, *M*:

$$x = (x_1, x_2, \ldots, x_K),$$
$$y = (y_1, y_2, \ldots, y_M).$$

Than production possibility set is defined as:

$$PPS = \{x, y : x \text{ can produce } y\}$$

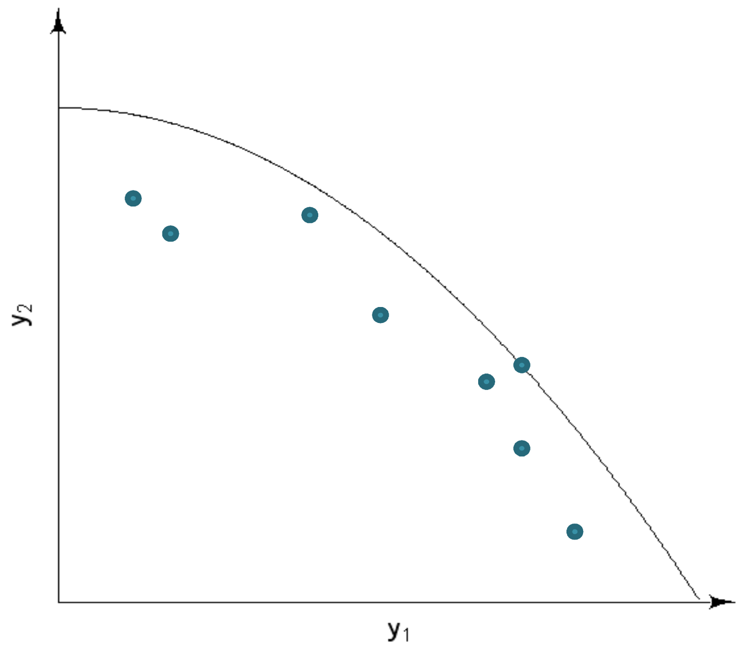
The set of feasible outputs for an input vector:

$$P(x) = \{y : (x, y) \in PPS\}$$

Production possibility frontier

A production possibility frontier is defined as a function:

$$f(x) = \{y : y \in P(x), \forall_{y'>y}\ y' \notin P(x)\}$$



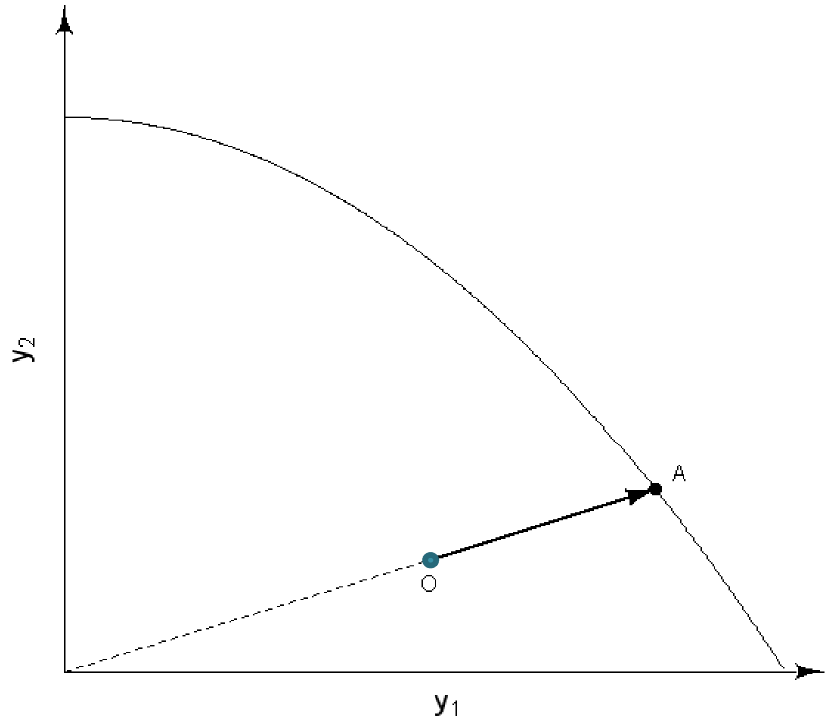
Technical efficiency

Debreu-Farrell definition of technical efficiency of output vector *y* is:

$$TE(x,y) = \left[\sup_{\theta} \{ \theta : \theta y \le f(x) \} \right]^{-1}$$

can be presented in a form of equation:

$$y = f(x) \cdot TE(x,y)$$



Classical stochastic frontier

For estimation purposes the technical efficiency term is usually transformed as:

$$TE(x, y) = \exp(-u), u \geq 0.$$

Introducing the random disturbances v into the formula, we consider a classical stochastic frontier model:

$$y_i = f(x_i, \beta) \cdot \exp(v_i) \cdot \exp(-u_i)$$

where β is a vector of unknown coefficients.

Classical stochastic frontier

The model is frequently presented in the logarithmic form:

$$\ln y_i = \ln f(x_i, \beta) + v_i - u_i$$

So the only probabilistic feature of the stochastic frontier model is a composed error term:

$$\varepsilon_i = v_i - u_i$$

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Stochastic frontier: a case of truncated normal inefficiency

The distribution of the random disturbances *v* is usually set to independent identically distributed normal with zero mean and constant deviation σ_v:

$$v_i \sim N\left(0, \sigma_v^2\right)$$

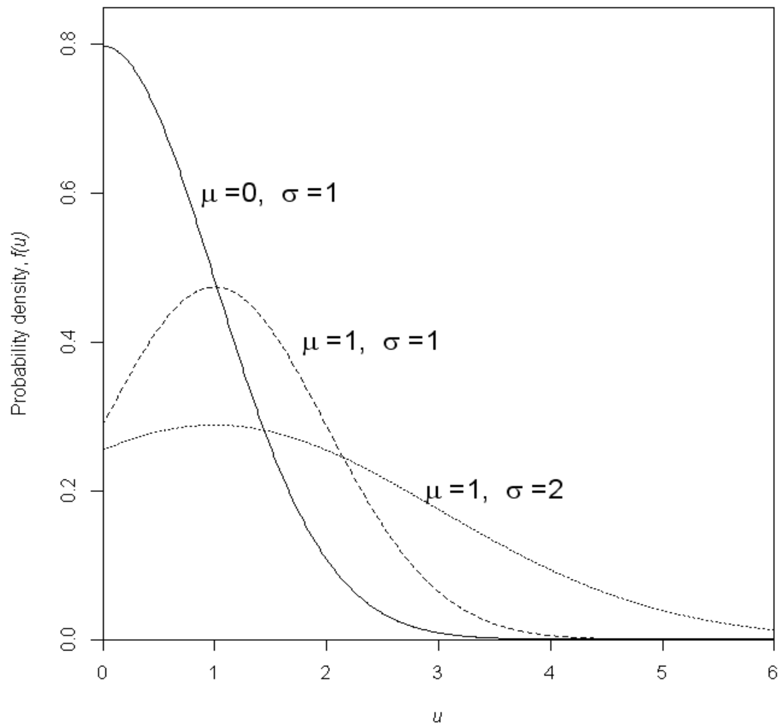
The inefficiency term u can be modelled with different distributions, e.g. truncated normal:

$$u_i \sim TN_{0,+\infty}\left(\mu, \sigma_u^2\right)$$

Stochastic frontier: a case of truncated normal inefficiency

The probability density function for truncated normal distribution is (truncation limits are set to 0 and +∞ to match the non-negativity requirement of u):

$$f(u_i) = \begin{cases} \left[\Phi\left(\frac{\mu}{\sigma_u}\right)\right]^{-1} \frac{1}{\sqrt{2\pi\sigma_u}} \exp\left(-\frac{(u_i-\mu)^2}{2\sigma_u^2}\right), u_i \geq 0 \\ 0, u_i < 0 \end{cases}$$



Stochastic frontier: a case of truncated normal inefficiency

According to the convolution formula, if u and v are independent, then:

$$\varepsilon_i = v_i - u_i$$

$$f_\varepsilon(\varepsilon_i) = \int_{-\infty}^{+\infty} f_v(\varepsilon_i + u_i) f_u(u_i) du_i$$

Stochastic frontier: a case of truncated normal inefficiency

Transformations:

$$f_\varepsilon(\varepsilon_i)=\int_{-\infty}^{+\infty} f_v(\varepsilon_i+u_i)f_u(u_i)du_i=$$

$$=\int_0^{+\infty}\left[\frac{1}{\sqrt{2\pi}\sigma_v}\exp\left(-\frac{(\varepsilon_i+u_i)^2}{2\sigma_v^2}\right)\right]\left[\left(\Phi\left(\frac{\mu}{\sigma_u}\right)\right)^{-1}\frac{1}{\sqrt{2\pi}\sigma_u}\exp\left(-\frac{(u_i-\mu)^2}{2\sigma_u^2}\right)\right]du_i=$$

$$=\left(\Phi\left(\frac{\mu}{\sigma_u}\right)\right)^{-1}\frac{1}{2\pi\sigma_v\sigma_u}\int_0^{+\infty}\exp\left(-\frac{(\varepsilon_i+u_i)^2}{2\sigma_v^2}-\frac{(u_i-\mu)^2}{2\sigma_u^2}\right)du_i=$$

$$=\left(\Phi\left(\frac{\mu}{\sigma_u}\right)\right)^{-1}\frac{1}{2\pi\sigma_v\sigma_u}\int_0^{+\infty}\exp\left(-\frac{1}{2}\left(\frac{\varepsilon_i+\mu}{\sqrt{\sigma_v^2+\sigma_u^2}}\right)^2\right)\exp\left(-\frac{1}{2}\left(-\frac{\varepsilon_i\sigma_u/\sigma_v}{\sqrt{\sigma_v^2+\sigma_u^2}}+\frac{\mu}{(\sigma_u/\sigma_v)\sqrt{\sigma_v^2+\sigma_u^2}}\right)^2\right)du_i=$$

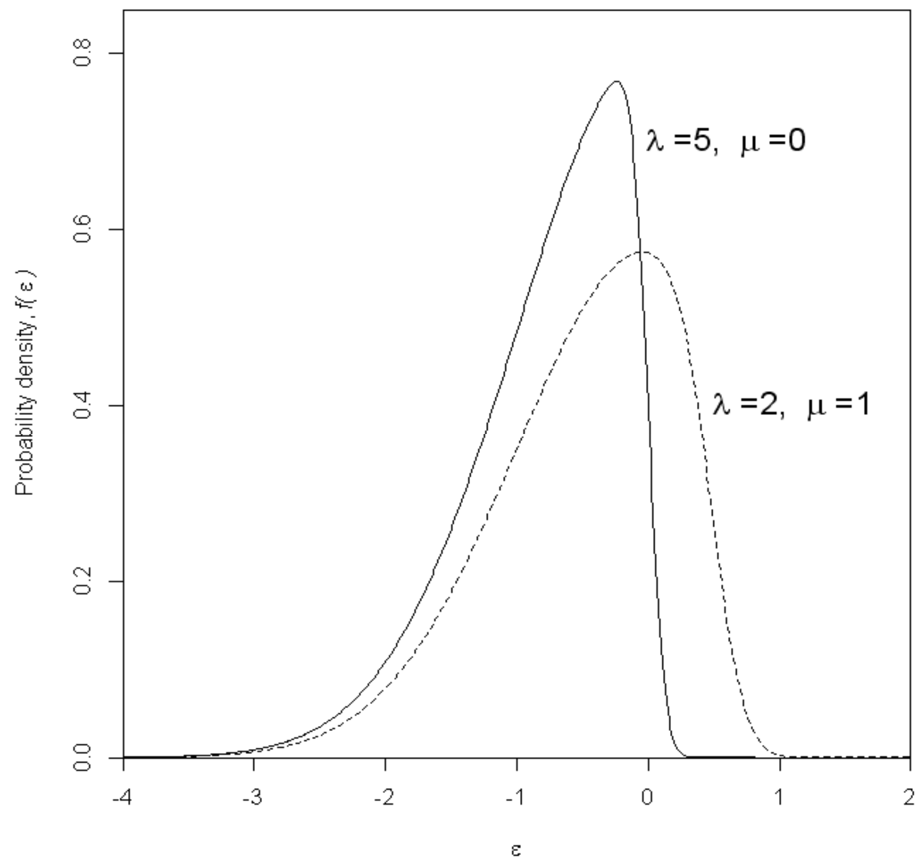
$$=\frac{1}{\sqrt{\sigma_v^2+\sigma_u^2}}\left(\Phi\left(\frac{\mu}{\sigma_u}\right)\right)^{-1}\varphi\left(\frac{\varepsilon_i+\mu}{\sqrt{\sigma_v^2+\sigma_u^2}}\right)\Phi\left(-\frac{\varepsilon_i\sigma_u/\sigma_v}{\sqrt{\sigma_v^2+\sigma_u^2}}+\frac{\mu}{(\sigma_u/\sigma_v)\sqrt{\sigma_v^2+\sigma_u^2}}\right)$$

Univariate Extended skew normal distribution

So

$$f_\varepsilon(\varepsilon_i) = \frac{1}{\sqrt{\sigma_v^2+\sigma_u^2}}\left(\Phi\left(\frac{\mu}{\sigma_u}\right)\right)^{-1}\varphi\left(\frac{\varepsilon_i+\mu}{\sqrt{\sigma_v^2+\sigma_u^2}}\right)\Phi\left(-\frac{\varepsilon_i\sigma_u/\sigma_v}{\sqrt{\sigma_v^2+\sigma_u^2}}+\frac{\mu}{(\sigma_u/\sigma_v)\sqrt{\sigma_v^2+\sigma_u^2}}\right)$$

This density function is known as an univariate **extended skew normal** distribution function, introduced by Azzalini (1985).



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Spatial stochastic frontier model

A basic assumption of the classical stochastic frontier:

Observations are independent!

$$y = X\beta + v - u,$$

$$v \sim MVN\left(0_n, \sigma_{\tilde{v}}^2 I_n\right),$$

$$u \sim MVTN_{0,+\infty}\left(\mu, \sigma_{\tilde{u}}^2 I_n\right)$$

In practice, there are a lot of theoretical and empirical evidences of relationships between observations.

Spatial stochastic frontier model

One of the simplest forms of this relationship is a linear combination:

$$y_i \text{ depends on } \sum_{i \neq j} w_{ij} y_j$$

In spatial econometrics this is assumed that coefficients w_{ij} can be explained with a distance between objects i and j, e.g.

$$w_{ij} = \frac{1}{\text{distance}\left(object_i, object_j\right)}$$

Spatial stochastic frontier model

Spatial contiguity matrix:

$$W = \begin{bmatrix} 0 & w_{12} & \dots & w_{1n} \\ w_{21} & 0 & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{n1} & w_{n2} & \dots & 0 \end{bmatrix}$$

Matrix form of dependence (spatial lags):

$$\forall_{i=1..n}\ y_i \text{ depends on } \sum_{i \neq j} w_{ij} y_j \Leftrightarrow y \text{ depends on } \underline{Wy}$$

Spatial stochastic frontier model

The spatial stochastic frontier model specification:

$$Y = \underline{\rho_Y W_Y Y} + X\beta + \underline{W_X X\beta^{(s)}} + v - u,$$
$$v = \underline{\rho_v W_v v} + \tilde{v},$$
$$u = \underline{\rho_u W_u u} + \tilde{u}.$$

where

- W_Y and ρ_Y are contiguity matrix and coefficient for endogenous spatial effects (spatial dependency),
- W_X and $\beta^{(s)}$ are contiguity matrix and coefficients for exogenous spatial effects,
- W_v and ρ_v are contiguity matrix and coefficients for spatially correlated random disturbances (spatial heterogeneity),
- W_u and ρ_u are contiguity matrix and coefficients for spatially related inefficiency.

Spatial stochastic frontier model

Assuming that $\tilde{v} \sim MVN\left(0_n, \sigma_{\tilde{v}}^2 I_n\right),$

$$\tilde{u} \sim MVTN_{0,+\infty}\left(\mu, \sigma_{\tilde{u}}^2 I_n\right)$$

and using a simple transformation

$$v = \left(I - \rho_v W_v\right)^{-1} \tilde{v},$$

$$u = \left(I - \rho_u W_u\right)^{-1} \tilde{u},$$

we obtain random component distributions:

$$v \sim MVN\left(0_n, \Sigma_v\right),$$

$$\Sigma_v = \sigma_{\tilde{v}}^2 \left(I - \rho_v W_v\right)^{-1} \left[\left(I - \rho_v W_v\right)^{-1}\right]^T$$

$$u \sim MVTN_{0,+\infty}\left(\mu, \Sigma_u\right),$$

$$\Sigma_u = \sigma_{\tilde{u}}^2 \left(I - \rho_u W_u\right)^{-1} \left[\left(I - \rho_u W_u\right)^{-1}\right]^T$$

Spatial stochastic frontier model

The main problem is a distribution of the composed error term:

$$\varepsilon = v - u$$

$$v \sim MVN\left(0_n, \Sigma_v\right), u \sim MVTN_{0,+\infty}\left(\mu, \Sigma_u\right).$$

Using of the convolution formula for derivation of the probability density function for ε is quite complicated (see in my PhD thesis).

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Moment generating and characteristic functions

Let *f(x)* is a continuous probability density function of a random variable X.

Moment generating function for a continuous probability density function:

$$MGF_X(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx$$

Characteristic function for a continuous probability density function:

$$CF_X(t) = \int_{-\infty}^{+\infty} e^{itx} f(x) dx \quad (i^2 = -1)$$

Moment generating and characteristic functions

Note that

$MGF_X(-t)$ is the **two-sided Laplace transformation** of the probability density function,

$CF_X(t)$ is an **inverse Fourier transformation** of the probability density function.

Moment generating and characteristic functions

Very useful properties of MGF:

1. **Uniqueness theorem**: if two distributions have the same moment-generating function, then they are identical at almost all points.

2. **Convolution theorem**: if random variables X and Y are independent and $Z=X+Y$, then:

$$MGF_Z(t)=MGF_X(t)MGF_Y(t)$$

Moment generating and characteristic functions

Classical stochastic frontier:

$$v_i \sim N\left(0, \sigma_v^2\right) \qquad MGF_{v_i}(t) = \exp\left(\frac{\sigma_v^2 t^2}{2}\right)$$

$$u_i \sim TN_{0,+\infty}\left(\mu, \sigma_u^2\right) \qquad MGF_{u_i}(t) = \exp\left(\mu t + \frac{\sigma_u^2 t^2}{2}\right)\left[\Phi\left(\frac{\mu}{\sigma_u}\right)\right]^{-1} \Phi\left(\frac{\mu}{\sigma_u} + \sigma_u t\right)$$

So the composed error term:

$$\varepsilon_i = v_i - u_i$$

$$MGF_{\varepsilon_i}(t) = -MGF_{v_i}(t) MGF_{u_i}(-t) =$$

$$= \exp\left(-\mu t + \frac{\left(\sigma_v^2 + \sigma_u^2\right) t^2}{2}\right)\left[\Phi\left(\frac{\mu}{\sigma_u}\right)\right]^{-1} \Phi\left(\frac{\mu}{\sigma_u} - \sigma_u t\right)$$

Moment generating and characteristic functions

Similarly for the spatial stochastic frontier model

$$v \sim MVN(0_n, \Sigma_v)$$

$$MGF_v(t) = \exp\left(\frac{1}{2} t^T \Sigma_v t\right)$$

$$u \sim MVTN_{0,+\infty}(\mu, \Sigma_u).$$

$$MGF_u(t) = [\Phi_n(0_n, -\mu, \Sigma_u)]^{-1} \exp\left(t^T + \frac{1}{2} t^T \Sigma_u t\right) \Phi_n(\Sigma_u t, -\mu, \Sigma_u)$$

Moment generating and characteristic functions

MGF of the composed error term:

$$\varepsilon_i = v_i - u_i$$

$$MGF_{\varepsilon_i}(t) = MGF_{v_i}(t) MGF_{u_i}(-t) =$$

$$= [\Phi_n(0, \mu, \Sigma_u)]^{-1} \exp\left(-t^T + \frac{1}{2} t^T (\Sigma_v + \Sigma_u) t\right) \Phi_n(-\Sigma_u t, -\mu, \Sigma_u)$$

which is a specific case of the **Closed Skew Normal distribution**.

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Closed Skew Normal distribution

The Closed Skew Normal distribution:

$$x \sim CSN_{n,n}(\mu', \Sigma', \Gamma', \nu', \Delta'),$$

$$\mu' = -\mu,$$

$$\Sigma' = \Sigma_v + \Sigma_u,$$

$$\Gamma' = -\Sigma_u (\Sigma_v + \Sigma_u)^{-1},$$

$$\nu' = -\mu,$$

$$\Delta' = \left(\Sigma_v^{-1} + \Sigma_u^{-1}\right)^{-1}$$

Introduced by González-Farías, Domínguez-Molina, and Gupta (2004).

Unified Skew Normal distribution

There are a lot of different variants of multivariate skewed distributions:

1. Closed Skew Normal
2. Hierarchical Skew Normal (Liseo, Loperfido, 2003)
3. Fundamental Skew Normal (Arellano-Valle, Genton, 2005)

Arellano-Valle and Azzalini (2006) introduced a **Unified Skew Normal distribution**, which cover many private cases, including CSN.

Estimation of the spatial stochastic frontier model

Maximum likelihood estimator:

$$\ln L\left(\beta, \beta^{(s)}, \sigma_{\tilde{v}}^2, \sigma_{\tilde{u}}^2, \mu, \rho_Y, \rho_v, \rho_u\right) = -\ln \Phi_n\left(0, -\mu, \Sigma_u\right) +$$

$$+ \ln \Phi_n\left(-\Sigma_u\left(\Sigma_v + \Sigma_u\right)^{-1}(e+\mu), -\mu, \left(\Sigma_v^{-1} + \Sigma_u^{-1}\right)^{-1}\right) + \ln \varphi_n\left(e, -\mu, \Sigma_v + \Sigma_u\right),$$

$$e = Y - \rho_Y W_Y Y - X\beta - W_X X\beta^{(s)},$$

$$\Sigma_v = \sigma_{\tilde{v}}^2\left(\left(I_n - \rho_v W_v\right)^{-1}\right)^T \left(I_n - \rho_v W_v\right)^{-1},$$

$$\Sigma_u = \sigma_{\tilde{u}}^2, \left(\left(I_n - \rho_u W_u\right)^{-1}\right)^T \left(I_n - \rho_u W_u\right)^{-1}.$$

Estimation of the spatial stochastic frontier model

Inefficiency component *u* can be estimated via conditional distribution, which is multivariate truncated normal:

$$u|\varepsilon \sim MVTN_{0,+\infty}\left(\mu_{u|\varepsilon}, \Sigma_{u|\varepsilon}\right),$$

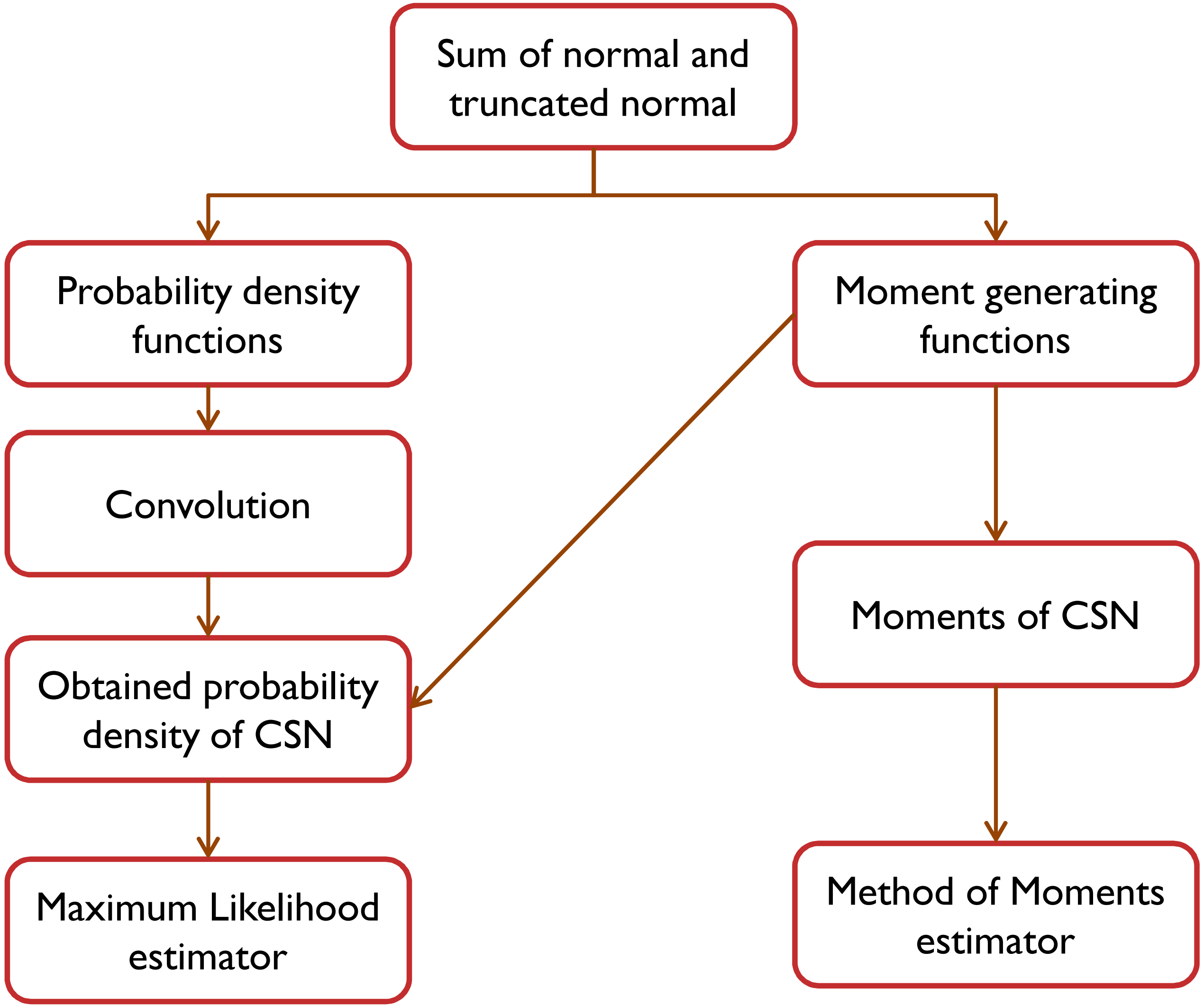
where

$$\mu_{u|\varepsilon} = \mu - \Sigma_u\left(\Sigma_v + \Sigma_u\right)^{-1}\left(\varepsilon + \mu\right)$$

$$\Sigma_{u|\varepsilon} = \left(\Sigma_v^{-1} + \Sigma_u^{-1}\right)^{-1}$$

Corresponding theoretical moments of the multivariate truncated normal distribution are well-known.

Alternative research paths



Related literature

1. González-Farías, G., Domínguez-Molina, J.A., Gupta, A. (2004). The closed skew-normal distribution, in *Skew-elliptical distributions and their applications: a journey beyond normality*, Chapman & Hall/CRC, Boca Raton, FL, pp. 25–42.
2. Domínguez-Molina, J.A., González-Farías, G., Ramos-Quiroga, R. (2004). Skew-normality in stochastic frontier analysis, *Skew-elliptical distributions and their applications: A journey beyond normality*, pp. 223–241.
3. Arellano-Valle, R. B. and Azzalini, A. (2006). On the unification of families of skew-normal distributions. Scand. J. Statist., 33, 561-574.
4. Aziz M. (2011) Study of Unified Multivariate Skew Normal Distribution with Applications in Finance and Actuarial Science, PhD thesis.

Thank you for your attention!

Questions are very appreciated