TESTING FOR CRACK INITIATION IN FATIGUE-SENSITIVE AIRCRAFT STRUCTURES WITH REPAIR UNDER TYPE I CENSORED SAMPLING

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Inferences concerning the location and scale parameters of a two-parameter exponential distribution of time to crack initiation due to fatigue, when crack is repaired, are discussed for samples censored at a fixed value. The procedures considered are uniformly most powerful unbiased if the tests are performed with repair. An example is given.

Keywords: f*atigue crack, repair, two-parameter exponential distribution, censored sampling*

1. INTRODUCTION

In this paper, we discuss the case of the testing with repair. Suppose that n items are tested, each having failure time with density function of the two-parameter exponential distribution,

$$f(x;\mu,\sigma) = (1/\sigma)\exp\{-(x-\mu)/\sigma\}, \quad \mu \le x < \infty, \sigma > 0, \tag{1}$$

and that these failure times are independent. This model is useful in life testing experiments; in these situations the location parameter μ is often called a guarantee or threshold parameter. In the case being considered, an item, which fails before the termination time, is repaired and tested again. We suppose that the repaired items have an exponential distribution with the same scale parameter σ as the original items but with location parameter 0. Since a repaired part would not be expected to perform as long as a new part, some assumption like this one would be appropriate. Even if defective parts were replaced, such an assumption might be reasonable if the original parts or system were sealed or treated in a special manner. This assumption was also used by Mann, Schafer, and Singpurwalla [1] to obtain the model they considered for testing with replacement for type I and type II censoring. Based on (possibly) censored data, Mann, Schafer, and Singpurwalla [1] discuss point estimates of μ and σ. They also present tests of hypotheses and confidence intervals for these parameters in the case of type II censoring (in which the number of failures is fixed) but not in the case of type I censoring (in which the duration of the experiment, as measured by time or some other appropriate quantity, is fixed). Such inferences based on type I censored data are discussed here for testing with repair. To be specific, we assume that the failures are occurring with time and that the duration of the experiment is some fixed time, say t units ($t > 0$). However, in the fatigue example discussed below, the failure times are measured in terms of load cycles. We also assume $\mu < t$ since there will be no failures otherwise and assume that the lifetimes of the original and repaired parts are mutually independent.

For a specific example, we consider the results of a fatigue test conducted by Butler and Rees [2] to determine the suitability of various metals for aircraft construction. In one phase of the study, titanium and steel specimens were tested for crack initiation due to fatigue. Each specimen was subjected to stresses in varying amounts and patterns similar to those occurring in flight. These stress patterns were repeated until a crack was detected, the number of load cycles (which can be thought of as a laboratory measure of flight time) until crack detection was recorded, the crack was repaired, the test was resumed until another crack was detected, the total number of load cycles to this failure was recorded, etc. The data given in their report (Butler and Rees [2]) definitely indicate that the number of load cycles before the first failure is considerably larger than the number of load cycles between successive failures. In the data considered here in Section 3, the number of load cycles to the first failure is about six times that of the largest successive difference. Also, the chi-squared goodness-of-fit test indicates that for this data the exponential distribution provides a reasonable model for the inter-failure times in the range that these tests were conducted.

In type I censoring the number of failures before time t is a random variable, which we denote by R. Because of the lack-of-memory property of an exponential distribution with location parameter μ, R has, in this case, a Poisson distribution with mean $n(t - \mu)/\sigma$. Given that $R = r > 0$, the ordered failure times $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ have the same distribution as the ordered values of a random sample from a uniform distribution on the interval (μ, t). Hence these ordered failure times and R have likelihood function

$$f\left(x_{(1)}, x_{(2)}, \ldots, x_{(r)}, r; \mu, \sigma, t, n\right) = f\left(x_{(1)}, x_{(2)}, \ldots, x_{(r)}; \mu, t, R = r\right) \Pr\{R = r; \mu, \sigma, t, n\}$$

$$= (n/\sigma)^r \exp[-n(t-\mu)/\sigma],$$

$$\mu < x_{(1)} < x_{(2)} < \ldots < x_{(r)} < t, \quad r = 1, 2, \ldots, \tag{2}$$

where

$$f\left(x_{(1)}, x_{(2)}, \ldots, x_{(r)}; \mu, t, R = r\right) = \frac{r!}{(t-\mu)^r}, \tag{3}$$

$$\Pr\{R = r; \mu, \sigma, t, n\} = \frac{[n(t-\mu)/\sigma]^r}{r!} \exp[-n(t-\mu)/\sigma], \tag{4}$$

and

$$\Pr\{R = 0\} = \exp[-n(t-\mu)/\sigma]. \tag{5}$$

Since R, the number of failures is random; the likelihood function has been written as a function with a variable number of arguments. Applying the factorization criterion for a family of distributions dominated by a σ-finite measure, we see that $X_{(1)}$ and R are jointly sufficient for (μ, σ), that $X_{(1)}$ is sufficient for μ with σ fixed, and that R is sufficient for σ with μ fixed (see Lehmann [3]). The dominating measure is defined on the union of $\{0\}$ and the sets $\{(x_{(1)}, \ldots, x_{(r)}, r): x_{(1)} < \ldots < x_{(r)} < t\}$ for $r = 1, 2, \ldots$). Although (2) is not a member of the regular exponential class, as shown in Section 2, the tests obtained from the appropriate conditional tests are uniformly most powerful unbiased and consequently, they give rise to optimal confidence limits.

2. TESTING WITH REPAIR

As noted in Section 1, $X_{(1)}$ and R are sufficient in the case being considered. For convenience we will define $X_{(1)}$ as the time of the first failure if this is less than t, and $X_{(1)} = t$ otherwise. First, optimal conditional tests will be developed and then shown to be optimal when viewed as unconditional tests for one of the parameters with the other considered as a nuisance parameter.

For inferences about σ with μ a nuisance parameter, we first need the conditional distribution of R given $X_{(1)}$. If $X_{(1)}=t$, then $R = 0$, and if $X_{(1)}=x_{(1)}<t$, then "restarting" the Poisson process at $x_{(1)}$, it is clear that $R-1$ has a Poisson distribution with mean $n(t - x_{(1)})/\sigma$. Since the density of $R-1$ is known to have a monotone likelihood ratio, the usual one-sided tests based on R are uniformly most powerful in this conditional framework. (If the uniformly most powerful unbiased two-sided test is preferred over the equal-tails test, then the rejection region must be chosen to satisfy the equality given in Lehmann [3]). In this procedure we randomize if $R = 0$. Note that in practice it may not be possible to obtain an exact α-level test since the test statistic R has a discrete distribution, so it may also be necessary to randomize for this reason.

For inferences about μ with σ viewed as a nuisance parameter, we obtain the conditional density of $X_{(1)}$ given R. If $R = 0$, then $X_{(1)}=t$, and if $R = r > 0$, then it is well-known that $X_{(1)}$ has the same distribution as the smallest order statistic of a random sample of size r taken from a uniform distribution on the interval (μ, t). So for $r > 0$ this conditional density is

$$f(x_{(1)};r) = r(t-\mu)^{-r}(t-x_{(1)})^{r-1}, \quad \mu < x_{(1)} < t. \tag{6}$$

Again this family of densities has a monotone likelihood ratio and the usual one-sided tests based on $X_{(1)}$ are uniformly most powerful in the conditional framework. The equal-tails test could be used in the two-sided case. It should be noted that given $R = r > 0$, $[(t-X_{(1)})/(t-\mu)]^r$ has a uniform distribution on the interval (0, 1). Since $[(t-x_{(1)})/(t-\mu)]^r$ is a decreasing function of $x_{(1)}$, inferences based on either $X_{(1)}$ or this pivotal quantity are equivalent but, of course, small values of $X_{(1)}$ correspond to large values of the pivotal quantity, and vice versa. Again this test is conditioned on $R > 0$.

The one-sided tests described in this section can also be viewed as unconditional tests. Appealing to Theorems 1 and 2 given by Ferguson [4], we see that these tests are uniformly most powerful unbiased provided

(i) every test of these hypotheses has power which is a continuous function of (μ,σ);
(ii) for each fixed μ, R is sufficient and complete for σ; and
(iii) for each fixed σ, $X_{(1)}$ is sufficient and complete for μ.

The required sufficiency of R and of $X_{(1)}$ follows by inspection of (2) and the completeness of R follows because it has a re-parameterized Poisson distribution. Next, $X_{(1)}$ is shown to be complete (with σ held fixed). Now $X_{(1)}$ has the same distribution as the smallest order statistic of a random sample of size n from (1) except that the probability that this order statistic exceeds t is assigned to t. That is, $X_{(1)}$ has a mixed discrete-absolutely continuous distribution with

$$\Pr\{X_{(1)} = t\} = \exp\{-n(t-\mu)/\sigma\} \tag{7}$$

and density

$$f(x) = (n/\sigma)\exp\{-n(x-\mu)/\sigma\} \quad \text{for} \quad \mu < x < t. \tag{8}$$

So for any function $h(\cdot)$, for which $E(h(X_{(1)})) = 0$ for all $\mu<t$

$$\int_\mu^t h(x)f(x)dx + h(t)\exp[-n(t-\mu)/\sigma] = 0, \tag{9}$$

and differentiating (9) with respect to μ, we see that $h(\cdot)$ is essentially zero.

Finally we establish (i). Denoting an arbitrary test function by $\varphi(r, x_{(1)}, ..., x_{(r)})$, we need only show that

$$g(\mu,\sigma) = \sum_{r=1}^{\infty}(n/\sigma)^r f(r,\mu), \tag{10}$$

where

$$f(r,\mu) = \int\limits_{\mu<x_{(1)}<...<x_{(r)}<t}\!\!\!\!\cdots\int \varphi(r, x_{(1)},...,x_{(r)})dx_{(1)}dx_{(2)}...dx_{(r)}, \tag{11}$$

is a continuous function of (μ,σ). Letting (μ',σ') denote another parameter point with $\mu'>\mu$, $|g(\mu,\sigma) - g(\mu',\sigma')|$ is bounded above by

$$\sum_{r=1}^{\infty}(n/\sigma')^r \,|\, f(r,\mu') - f(r,\mu)\,| + \sum_{r=1}^{\infty} f(r,\mu)\,|\,(n/\sigma')^r - (n/\sigma)^r\,|. \tag{12}$$

To establish the desired continuity we bound both expressions in (12) by quantities, which converge to zero as $|\mu-\mu'|$ and $|\sigma-\sigma'|$ converge to zero.

Since $0 \le \varphi(\cdot) \le 1$, $0 \le f(r,\mu) \le (t-\mu)^r/r!$ and by the mean value theorem $\left|(n/\sigma')^r - (n/\sigma)^r\right| \le r\left|n/\sigma' - n/\sigma\right|(n/\sigma^\bullet)^{r-1}$, where $\sigma^\bullet = \min(\sigma,\sigma')$. Applying these two bounds to the second expression in (12), we see that it is bounded above by

$$(t-\mu)\left|n/\sigma' - n/\sigma\right|[\exp[n(t-\mu)/\sigma^\bullet]. \tag{13}$$

Since $|\,f(r,\mu') - f(r,\mu)\,|$ is bounded above by

$$\int_\mu^{\mu'} (t - x_{(1)})^{r-1}/(r-1)!dx_{(1)} \le |\,\mu' - \mu\,|\,(t-\mu)^{r-1}/(r-1)!, \tag{14}$$

the first expression in (12) is bounded by

$$(n/\sigma')\,|\,\mu' - \mu\,|\exp[n(t-\mu)/\sigma']. \tag{15}$$

Thus, these one-sided tests are uniformly most powerful unbiased and consequently the associated confidence limits are uniformly most accurate unbiased.

3. EXAMPLE

To illustrate the results for testing with repair we consider some data from the fatigue-testing example discussed in the introduction. Recall that specimens of titanium and steel were subjected to stresses similar to those occurring in flight until a crack was detected; the crack was repaired, and testing was then resumed. The total "time" to detection of each crack was recorded in load cycles. For the tests discussed in Butler and Rees [2], no two specimens of the same metal were tested at the same temperature and humidity level, so we consider only the results from the test on one specimen.

Table 1 contains the total number of load cycles for the successive crack initiations for the first steel specimen with a termination "time" of 60,000 load cycles. Ties in the data may be attributed to several factors; for example, cracks beginning to form might be detected while others are being repaired. Note that the first crack was detected at 35,143 load cycles and the largest of the next twenty times between

Table 1. Total Number of Load Cycles to Successive Crack Detections

Crack no.	Load cycles	Crack no.	Load cycles	Crack no.	Load cycles
1	35,143	8	50,750	15	54,111
2	41,250	9	50,750	16	54,940
3	43,150	10	50,750	17	55,600
4	46,950	11	52,052	18	56,750
5	47,900	12	52,646	19	58,000
6	48,750	13	52,850	20	59,506
7	50,250	14	53,476	21	60,000

failures was 6,107 load cycles. This data set, as well as the other seven presented by Butler and Rees indicates that a model should be chosen which tends to have longer times to the first failure than the times between successive failures. The inclusion of the threshold parameter μ provides such a model. A chi-squared goodness-of-fit test was performed on the 20 times between failures (not including the time to the first failure) to determine if the exponential assumption is reasonable. The chi-squared value with two degrees of freedom was 2.00, so the assumption of exponential seems reasonable.

First we test H_0: $\mu = 0$ versus H_1: $\mu > 0$. The conditional test rejects if

$$\left(\frac{t - X_{(1)}}{t}\right)^r < \alpha \ , \tag{16}$$

and for this example

$$\left(\frac{t - X_{(1)}}{t}\right)^r = 9.19 \times 10^{-9}. \tag{17}$$

Therefore, under the distribution assumptions of Section 2, the inclusion of a threshold parameter is definitely justified, and we will obtain a lower confidence bound for σ with μ a nuisance parameter. We wish to determine σ_L as a function of r, which satisfies

$$\Pr\{R^{\bullet} \geq r - 1\} = 1 - \alpha, \tag{18}$$

where $R^{\bullet}$ is a Poisson variable with mean $n(t - x_{(1)})/\sigma_L$. However,

$$\Pr[R^{\bullet} \geq r - 1] = G(2n(t - x_{(1)})/\sigma_L; 2(r-1)), \tag{19}$$

where G(x;k) denotes the cumulative distribution function of a chi-squared variable with k degrees of freedom. Denoting the 100(1$-\alpha$)th percentile of such a distribution by $\chi^2(1-\alpha;\ k)$, the desired lower confidence bound is given by

$$\sigma_L = 2n(t - x_{(1)})/\chi^2(1 - \alpha; 2(r-1)). \tag{20}$$

In the example considered, r=21, n=l, t= 60,000, and $x_{(1)}$= 35,143, so a 95 percent lower confidence bound for σ is σ_L = 891.65.

4. CONCLUSION

The purpose of the paper has been to present inferences concerning the location and scale parameters of an exponential distribution under type I censored sampling. It is the authors' hope that this study will stimulate others to study the problem of testing for crack initiation due to fatigue in fatigue- sensitive structures with repair or without repair under type I censored sampling.

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