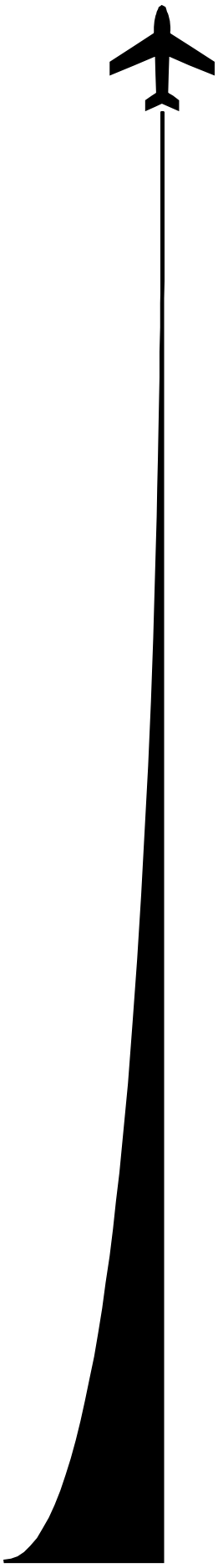

Session 2

Statistical Applications

PARAMETER ESTIMATION AND HYPOTHESES TESTING FOR NONHOMOGENEOUS POISSON PROCESS: PART 2. NUMERICAL EXAMPLES

I. B. Frenkel[1], I. B. Gertsbakh[2] and L. V. Khvatskin[1]*

[1] *Industrial Engineering and Management Department,*
Negev Academic College of Engineering
Beer-Sheva, 84100, Israel, E-mail: iliaf@nace.ac.il/ khvat@nace.ac.il
** Corresponding author. Tel.: +972-8-6475676, Fax: +972-8-6475643*
E-mail address: iliaf@nace.ac.il

[2] *Department of Mathematics, Ben-Gurion University,*
P. O. Box 653, Beer-Sheva, 84105, Israel,
E-mail: eliahu@cs.bgu.ac.il

Abstract

We consider data processing for a Nonhomogeneous Poisson process (NHPP) with Log-linear and Power-law intensity functions. Parameter estimation is carried out by the maximum likelihood method.

For the case of the known intensity function, testing the hypothesis that the given sample path is a realization of NHPP, can be accomplished using the fact, that under the NHPP model the mean value functions of NHPP, computed in sequence of ordered failure times, are the failure times of Homogeneous Poisson Process (HPP) with constant intensity function of one, and the intervals between events in the HPP form a sample of i.i.d. standard exponential random variables. Thus it is possible to use standard goodness-of fit tests to check the exponentiality of the process.

The computer-intensive procedure for testing the hypothesis that the given sample path belongs to NHPP without making the assumption that the intensity function is known was described in our previous article (Frenkel at al. (2003)).

In this article we describe the different goodness-of fit tests and demonstrate our method on the failure data which exist in literature and for our own failure data for the Schlosser Vibration Machine. We also demonstrate the several methods for generating families of stochastic processes with the known probabilistic structure, which includes both NHPP and not NHPP, and check how our method recognizes the underlying process. These processes were used for testing of power properties of goodness-of-fit tests.

Key words: Nonhomogeneous Poisson process, computer-intensive procedure, goodness-of-fit tests, hypotheses testing, comparison of powers of tests

1. Introduction

Nonhomogeneous Poisson Process (NHPP) is used to model reliability of repairable systems. There are many results on estimation and hypotheses testing concerning parameters of the intensity function of NHPP (Ascher and Feingold (1984), Crow (1974), Crowder et al. (1991), Gertsbakh (2000) and others).

Basing on failure data, it is necessary to develop tests that the data actually came from NHPP.

A commonly used procedure, as far as the reliability of a repairable system is concerned, is to distinct between Homogeneous Poisson Process (HPP) and NHPP. Cox and Lewis (1966) suggest the Laplace statistic and show that this test is optimal for testing NHPP with log-linear intensity function. Crowder et al. (1991) recommend the MIL-HDBK-189 (1981) test for testing NHPP with power-law intensity function.

Crow (1984) considers a goodness-of-fit (GOF) test, proving that the data came from NHPP via Cramer-von Mises statistic and obtained a table of critical values for this statistic. Park and Kim (1992) use the Kolmogorov-Smirnov, Cramer-von Mises and Anderson-Darling

statistics for GOF test of a power-law NHPP in the case of failure-truncated data. They present more precise tables of critical values for these statistics.

A simple GOF test for power-law NHPP, based on the Duane plot, is introduced by Gaudoin et al. (2003). It is based on graphical analysis of log-log plot of cumulative numbers of failures versus time. The authors claim that when a plot is completed, the coefficient of determination is used as the statistic of GOF test for NHPP. However, Ascher and Feingold (1984) at page 78 note that "plotting on log-log paper will tend to linearize any function. Hence a linear plot on such paper does not necessarily indicate that an NHPP … is the most appropriate model".

Many useful tests for testing of the assumption that the underlying distribution of life is exponential were introduced by Epstein (1960). Among others, a test of exponentiality is based on some results of Hartley (1950). This test deals with the ratio of maximum value to minimum value of the time intervals between failures. Gnedenko at al. (1969) suggest, using the Hartley's test, among othe.

In our previous article (Frenkel at al. (2003) we proposed a computer-intensive procedure for testing the hypothesis that the given sample path belongs to NHPP.

We suggest three kinds of data for demonstrating our method.

1. Using the existing in literature failure data, we compare our conclusions to other authors' results. Generally the results coincide.

2. We present a numerical example of our own data to show how the above described procedure functions.

3. We generate a set of stochastic processes with known probabilistic structure. This set includes NHPP and not NHPP. And we investigate how our method recognizes the underlying process and compare the powers of different tests.

2. Hypothesis Testing

2.1. Testing Hypothesis for NHPP

The procedure for estimation of NHPP intensity function parameters is described in literature (Cox and Lewis (1966)). Two parameterizations are especially convenient for maximum likelihood estimation: the log-linear form with $\lambda(t) = e^{\alpha+\beta t}$ and Weibull or power-law form with $\lambda(t) = \alpha^{\beta}\beta t^{\beta-1}$. The maximum likelihood estimators of (α, β) are obtained as the solutions of the systems of the ML equations (Crowder et al. (1991) pp. 167 and 173).

Testing the hypothesis, that the given process is NHPP with known intensity function $\lambda(t)$ is based on the claim, that the mean value functions of NHPP, counted in sequence of ordered failure times, are the failure times of Homogeneous Poisson Process (HPP) with constant intensity function of one, and the intervals between events in the HPP form a sample of i.i.d. standard exponential random variables (Çinlar (1975), Thompson (1981), Meeker and Escobar (1998)).

Thus it is possible to use standard goodness-of fit tests to check the exponentiality of the process.

The procedure for testing the hypothesis that the given sample path belongs to NHPP without making an assumption that the intensity function is known was described in our previous article (Frenkel at al. (2003)).

We used the following statistics notations for hypothesis testing. All statistic calculations will be explain in details in the next chapter.

S_1 – Kolmogorov-Smirnov distance between the ordered sample of in-between-events intervals $(\hat{\Delta}_1^{(i)}, \hat{\Delta}_2^{(i)}, ..., \hat{\Delta}_n^{(i)})$ in transformed time and the theoretical CDF, assuming that $\Delta_i \sim Exp(1)$;

S_2 – the Laplace statistic, also based on the in-between-events intervals $(\hat{\Delta}_1^{(i)}, \hat{\Delta}_2^{(i)}, ..., \hat{\Delta}_n^{(i)})$ in transformed time (Ascher and Feingold (1984), pp. 78-79, Crowder et al. (1991), p. 169);

S_3 – the coefficient of variation (C.V.) (Gertsbakh (2000), p. 214, Ascher and Feingold (1984), p. 82). C.V. is defined as the ratio between a standard deviation and mean value of random variable.

S_4 – The Levis-Robinson Statistic (Ascher and Feingold (1984), p. 82).

S_5 – MIL-HDBK-Test Statistic (Ascher and Feingold (1984), p. 79, Crowder et al. (1991), p. 172-173, MIL-HDBK-189 (1981)).

S_6 – The Cramer-von Mises Statistic (Park and Kim (1992)).

S_7 – The Anderson-Darling Statistic (Park and Kim (1992)).

S_8 – The Hartley Statistic (Gnedenko at al. (1969), p. 252).

After carrying out the procedure, we will obtain the simulated values of all above statistics. Determine the upper and lower α - critical values for these statistics and intervals $\left[S_i(\alpha), S_i(1-\alpha)\right], i = 1, 2, ..., k..$

For the given sample compute the values $S_1^*, S_2^*, ..., S_k^*$ of the statistics $S_1, S_2, \ldots, S_k$. Compare $S_1^*, S_2^*, ..., S_k^*$ with the upper and lower critical values.

Reject the hypothesis that the given process is the NHPP with intensity function $\hat{\lambda}(t)$, if one of the statistics $S_1^*, S_2^*, ..., S_k^*$ falls outside of the one of the intervals $\left[S_1(\alpha), S_1(1-\alpha)\right]$, $\left[S_2(\alpha), S_2(1-\alpha)\right], ..., \left[S_k(\alpha), S_k(1-\alpha)\right]$.

2.2. Statistics for Hypothesis Testing

2.2.1. The Laplace Trend Test

The Laplace's trend test is a simple and powerful test for distinguishing between Homogeneous Poisson Process, when a constant rate at which events are occurring, and an increasing rate of occurrence of such event. Laplace's Test is discussed in details in Cox and Lewis (1966), Ascher and Feingold (1984).

Consider a situation where the system is run until m failures have occurred. Under HPP assumption, the first $m-1$ arrival times, designated as $T_1, T_2, \ldots, T_{m-1}$, are the order statistics from a uniform distribution on [0, T_m] interval. The Laplace's test statistic is defined as

$$U = \frac{\dfrac{\sum_{i=1}^{m-1} T_i}{m-1} - \dfrac{T_m}{2}}{T_m \cdot \sqrt{\dfrac{1}{12(m-1)}}}.$$

The Laplace test statistic has the following interpretation. Under constant rate of event occurrence, the arrival times to fault will occur randomly around the midpoint of the length, $T_m/2$. Therefore, the sample mean of the T_i's will be approximately equal to $T_m/2$; hence the test statistic U will be small. If events are occurring more frequently towards the end of the interval, however, the sample mean will be large. If U L is larger than the z-value of the

standard normal distribution, $z_{\alpha/2}$, there is evidence at a significant level α that the event occurrence indicates the increasing trend.

2.2.2. The Lewis-Robinson Test for Trend

Meeker and Escobar (1998) note that both Laplace's Test and MIL-HDBK-189 Test can give misleading results when the renewal process is not an HPP. The Lewis-Robinson statistic (Lewis and Robinson (1974)) is formed by dividing the Laplace's statistic U by the estimated coefficient of variation of the times between recurrences,

$$U_{LR} = \frac{U}{C\hat{V}}.$$

If the underlying process is an HPP, then U_{LR} is asymptotically equivalent to U since CV=1 if the times between recurrences are exponentially distributed. In large samples, U_{LR} follow approximately a standard normal distribution if the underlying process is a renewal process.

Lawless and Thiagarajah (1966) indicate that the Lewis-Robinson Test is preferable to Laplace's test as a general test of trend in point process data.

The Lewis-Robinson Test was discussed by Ascher and Faingold (1984).

2.2.3. The MIL-HDBK-189 Test

The MIL-HDBK-189 (1981) test is based on the test statistic

$$V = 2\sum_{i=1}^{m-1} \ln\left[\frac{T_m}{T_i}\right].$$

Under the null Hypothesis of an HPP, V is distributed as χ^2, with 2(m-1) degrees of freedom. Large values of V supply evidence against null hypothesis in favor of reliability growth. Small values of V are indicative of reliability deterioration.

The MIL-HDBK-189 test was discussed by Crowder et al. (1991), Meeker and Escobar (1998) and others.

2.2.4. Cramer-von Mises Test

The failure times, T_i/T_m, i=1, 2, …, m-1 are distributed as order statistics from the uniform distribution over [0,1] (Park and Kim (1992)). Denote $\hat{U}_i = T_i/T_m$. Cramer-von Mises statistic calculates as follows:

$$W^2 = \sum_{i=1}^{n-1}\left[\hat{U}_i - \frac{2i-1}{2(n-1)}\right]^2 + \frac{1}{12(n-1)}.$$

Critical values of this goodness-of-fit statistic are calculated by Crow (1964, 1990). Park and Kim (1992) present more precise table of critical values for this statistic.

2.2.5. Anderson-Darling Test

Anderson-Darling statistic is calculated as follows (Park and Kim, 1992):

$$A^2 = -\frac{\sum_{i=1}^{n-1}(2i-1)\left[\ln\hat{U}_i + \ln(1-\hat{U}_{n-i})\right]}{n-1} - n + 1.$$

Critical values of this goodness-of-fit statistic are calculated by Park and Kim (1992).

2.2.6. Hartley Test

This test of exponentiality is based on some results of Hartley [12]. The test uses ratio of maximum value to minimum value of the time intervals between failures. Gnedenko at al. [11] suggest, among other tests, using the Hartley's test. In our case the Hartley's Test is as follows:

$$h(n) = \frac{\max\{\Delta_i\}}{\min\{\Delta_i\}}.$$

The null hypothesis will be rejected if $h(n) > h_{1-\alpha}(n)$. The critical values for this statistic are represented in Gnedenko at al. (1969) and for big n values (n>12) may be calculated using Monte-Carlo Simulation.

3. Testing the NHPP Hypothesis on Experimental and Simulated Data

3.1. Hypothesis Testing on Literature Failure Data

Let us consider the well-known real-life failure data and compare different authors' conclusions with results, gained via our procedure.

3.1.1. Proschan Data on Air Conditioning System

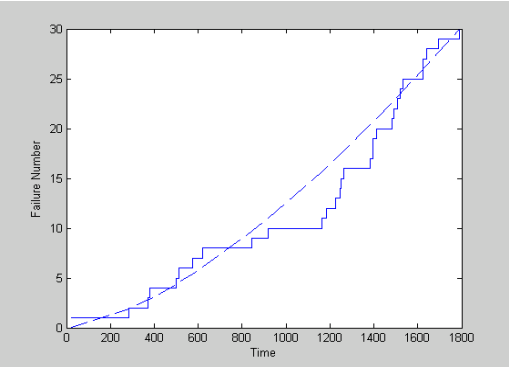

***Figure 1**. The Plot of the step function of the failure numbers and corresponding estimated intensity function against time for the air conditioning system*

Let us illustrate our methodology using data on the time intervals between successive failures of the air conditioning system of the Boeing 720 jet series number 7912 (Proschan (1963)). This data was analyzed by many researchers. Park and Kim (1992) define 3 statistics (Cramer-von Mises, Kolmogorov-Smirnov and Anderson-Darling) for the goodness-of-fit (GOF) test of this process. The authors claim, that failure data came indeed from NHPP with power-law intensity function. Gaudoin et al. (2003) introduce the graphical analysis for GOF test of the power-law process and received similar results. The same result was achived by Muralidharan (2001), Asher and Hansen (1998) and others.

We came to the similar conclusion (Table 1). All test statistic values fall inside the corresponding [0.05, 0.95] simulated intervals for all of our statistics. Therefore, we would claim that the data don't contradict the NHPP with power-law intensity function. On Figure 1 is shown the Plot of the step function of the failure numbers and corresponding estimated intensity function against time for the air conditioning system

Table 1. Comparison of Test Statistics with the Corresponding Simulated Critical Values for the Air Conditioning System

Test Statistic	Test Statistic Values	Corresponding Simulated Intervals	
		0.05	0.95
Kolmogorov-Smirnov Statistic	0.169	0.050	0.171
Laplace Statistic	1.069	-0.350	1.226
CV Statistic	1.140	0.732	1.239
Levis-Robinson Statistic	0.983	-0.352	1.323
MIL-HDBK-Statistic	60.000	59.263	60.600
Cramer-von Mises Statistic	0.140	0.029	0.228
Anderson-Darling Statistic	0.794	0.227	1.350
Hartley Statistic	196.197	34.288	2213.6

3.1.2. USS Halfbeak Engine Failures Data

Crowder et al. (1991) at the page 173 gives the data on failures of an engine of USS Halfbeak. The data were fitted using the Log-Linear and Power-Law intensity functions with the total number of observed events n=71. Using the Laplace test statistic and MIL-HDBK test statistic the authors express doubts that the data set comes from a NHPP.

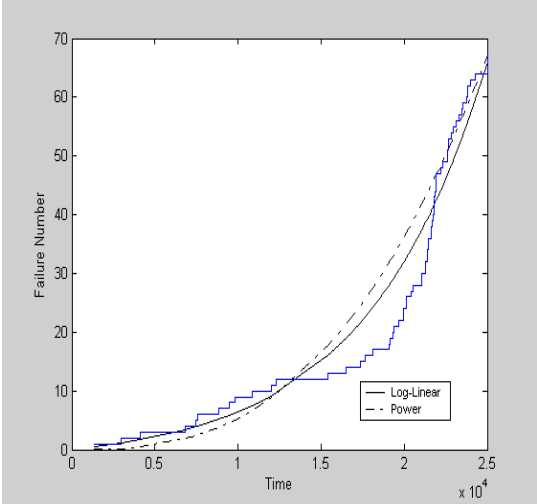

Figure 2. *The Plot of the step function of the failure numbers and corresponding estimated intensity function against time for the engine of USS Halfbeak*

Our tests reveal the following: Using the Power-Law intensity function four of eight statistics fall *outside* the corresponding [0.05, 0.95] simulated intervals for these statistics (Table 2). Using Log-Linear intensity function all our criteria do not contradict NHPP hypothesis. Our conclusion is that NHPP hypothesis is doubtful. On the Figure 2 is shown the Plot of the step function of the failure numbers and corresponding estimated intensity function against time for the engine of USS Halfbeak

Table 2. Comparison of Test Statistics with the Corresponding Simulated Critical Values for the engine of USS Halfbeak

Test Statistic	Log-Linear Intensity Function			Power-Law Intensity Function		
	Test Statistic Values	Corresponding Simulated Intervals		Test Statistic Values	Corresponding Simulated Intervals	
		0.05	0.95		0.05	0.95
Kolmogorov-Smirnov Statistic	0.150	0.061	0.202	**0.204**	0.055	0.129
Laplace Statistic	0.783	-2.149	2.559	**1.904**	-0.593	1.123
CV Statistic	1.228	0.832	1.312	1.34	0.81	1.149
Levis-Robinson Statistic	0.638	-2.090	2.543	**1.421**	-0.620	1.127
MIL-HDBK-Statistic	135.1	105.8	190.3	142.0	141.57	142.39
Cramer-von Mises Statistic	0.375	0.040	0.949	**0.680**	0.029	0.231
Anderson-Darling Statistic	1.808	0.301	4.913	**3.222**	0.225	1.373
Hartley Statistic	1130.5	113.2	5734.3	1463.4	105.9	6909.0

3.1.4. Schlosser Vibration Machine Failures Data

The data presented in Table 2 summarize the time-intervals in operating hours between failures of Schlosser Vibration Machine collected from the operation reports from 1999 to 2002 at the Yeroham Construction Materials Facility (Israel). The machine was observed to be operated for 16309 hours and 27 failures were identified.

Table 3. Schlosser Vibration Machine Failure Data

240	2352	888	88	456
4032	168	768	268	24
288	480	336	84	120
1224	1400	528	86	
624	408	72	96	
552	528	96	103	

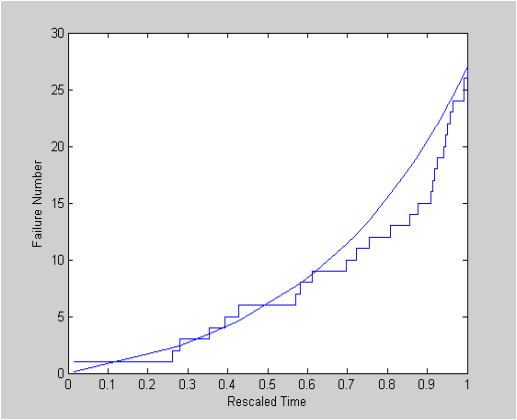

Figure 3. *The Plot of the step function of the failure numbers and corresponding estimated intensity function against rescaled time for the Schlosser Vibration Machine*

The estimated intensity function is assumed to be log-linear. Figure 3 shows the step function of the failure numbers against time in operation hours and corresponding estimated intensity function. We rescaled the failure time by a factor 16309. Hence α=1.7992 and β=2.4979.

To gain this failure data we used our method. According to Table 4 all test statistic values fall inside the corresponding simulated intervals. Therefore, we would claim that the data belong from NHPP with log-linear intensity function.

Table 4. Comparison of Test Statistics with Corresponding Simulated Intervals for the Schlosser Vibration Machine

Test Statistic	Test Statistic Values	Corresponding Simulated Intervals	
		0.05	0.95
Kolmogorov-Smirnov Test Statistic	0.075	0.070	0.507
Laplace Test Statistic	1.614	-2.180	3.002
CV Statistic	0.870	0.729	1.384
Levis-Robinson Test Statistic	1.854	-1.878	2.890
MIL-HDBK-Test Statistic	50.389	31.398	89.614
Cramer-von Mises Test Statistic	0.175	0.042	1.055
Anderson-Darling Test Statistic	0.984	0.320	5.555
Hartley Test Statistic	33.632	31.200	2221.4

3.3. Testing the Data with Known Structure

So far, we applied our testing procedure to the realizations of random processes with unknown structure. In some cases we may only suppose, using some physical considerations that the random path we have analyzed comes from NHPP. In order to show how our testing procedure may distinguishes between NHPP and not NHPP, we must simulate random processes with known probabilistic structure.

We suggest three methods to generate the random processes with known probabilistic structure. The first method is to fill up the random process with elements of two different known structures: exponential random variables and slightly disturbed constant value according the probability. The second method is to form the elements of random process as "mixture" of an exponential random variables and a constant value. And the third one is to slightly disturb the NHPP using non-homogeneous deterministic process.

3.3.1. The first method

Consider two urns with random numbers Ξ^1 and Ξ^2. The first urn consists of random numbers $\xi^1 \sim Exp(1)$. The second urn consists of the random numbers $\xi^2 \sim U(1-\varepsilon, 1+\varepsilon)$, where $\varepsilon = 0.01$ for example.

Make the following procedure.

1. Set the choice probability $\alpha = 0$.

2. Generate the random number $\beta \sim U(0,1)$.

3. If $\beta < 1-\alpha$, then draw a random number ξ from the first urn, otherwise, draw a random number from the second urn.

4. Repeat the process *n* times.

5. Set $\alpha = \alpha + 0.01$.

6. If $\alpha \le 1$, then go to 2.

We received a set of renewal processes $\xi_1, \xi_2, ..., \xi_n$. For $\alpha = 0$, we have a NHPP (in fact, a HPP) with $\lambda(t) = 1$ and for $\alpha = 1$, we have an "almost" deterministic sequence, which in some sense is opposite to NHPP. Using our procedure on this set of renewal processes we calculated the power of described above tests. The main results are shown in the Figures 4 and 5. Tests that not shown, are not sensitive for such processes.

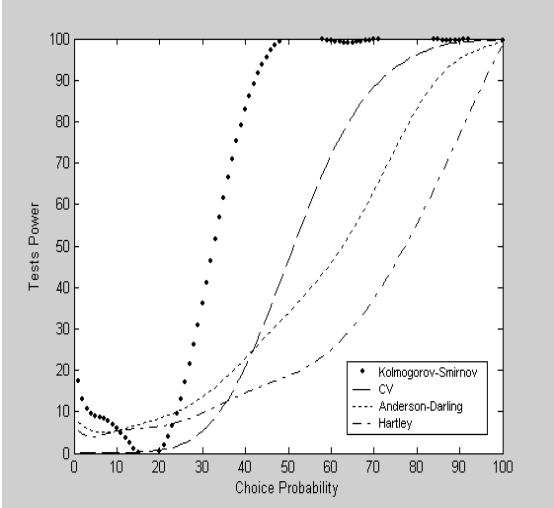

Figure 4. *Comparison of Tests Power with Log-Linear Function Recognition*

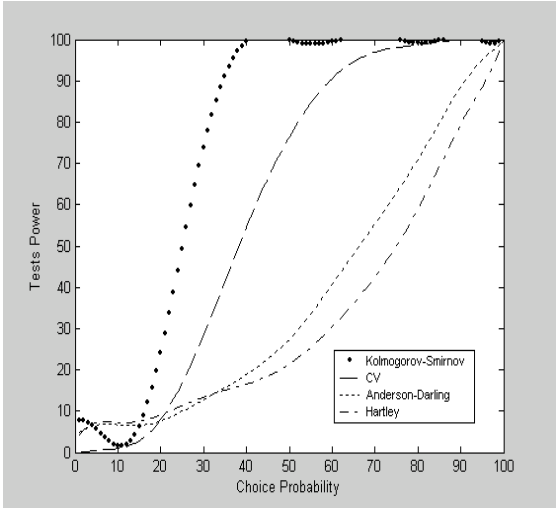

Figure 5*. Comparison of Tests Power with Power Function Recognition*

3.3.2. The second method

Consider two same urns with random numbers Ξ^1 and Ξ^2. The first urn consists of random numbers $\xi^1 \sim Exp(1)$. The second urn consists of the random numbers $\xi^2 \sim U(1-\varepsilon, 1+\varepsilon)$, where $\varepsilon = 0.01$ for example.

Set the α. Let us take the random number ξ^1 from the first urn and ξ^2 from the second. Make the linear combination

$$\xi = (1-\alpha)\xi^1 + \alpha\xi^2 .$$

Repeat the procedure *n* times. Receive the sequence of random inter invents intervals $\xi_1, \xi_2, ..., \xi_n$. Changing the parameter α from 0 to 1, we receive collection of renewal processes, where the first process for $\alpha = 0$ is a renewal process with standard exponentially distributed inter event times (NHPP with $\lambda(t) \equiv 1$,in fact, a HPP) and the last process for $\alpha = 1$ is a deterministic sequence. Using our procedure on this set of renewal processes we calculated the power of described above tests. The main results are shown in the Figures 6 and 7. Tests that not shown, are not sensitive for such processes.

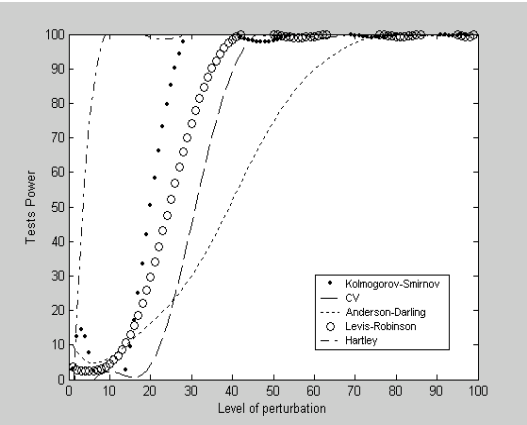

Figure 6. *Comparison of Tests Power with Log-Linear Function Recognition*

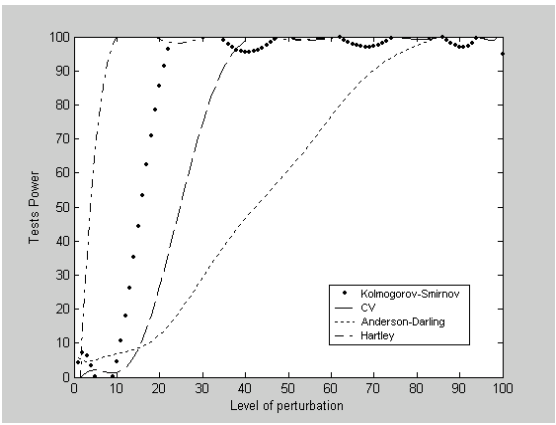

Figure 7. *Comparison of Tests Power with Power Function Recognition*

The sequence of purely exponential random variables is recognized by our tests as a HPP. Slightly perturbed sequence, e.g. $\alpha = 0.1 \div 0.2$, of exponential random variables are recognized by our tests as being HPP. Other processes are recognized by our tests as not being HPP.

3.3.3. The third method

Let us generate n evens of NHPP with intensity function $\lambda(t) = t$ using the reverse transformation of Exp(1) distributed sequence of random numbers.

1. Generate random numbers $x_i \sim Exp(1), \quad i = 1, 2, ..., n$
2. Compute $y_1 = x_1,\ y_2 = x_1 + x_2\ ,...,\ y_{100} = \sum_{i=1}^{100} x_i$
3. Define $\upsilon \equiv \lambda(\upsilon)$
4. Let $y_k = \int_0^{t_k} \upsilon d\upsilon = \frac{t_k^2}{2}, \quad k = 1, 2, ..., n$
5. Compute $t_k = \sqrt{2y_k}, \quad k = 1, 2, ..., n$

In our generated NHPP the events occur at the time moments

$$t_1 = \sqrt{2y_1}\ ,...,\ t_k = \sqrt{2y_k}\ ,...,\ t_n = \sqrt{2y_n}$$

and intervals between events are equal to

$$\delta_1^1 = t_1,\ \delta_2^1 = t_2 - t_1,\ ...\ , \delta_k^1 = t_k - t_{k-1},\ ...\ , \delta_n^1 = t_n - t_{n-1}.$$

Check this NHPP using our procedure with Power intensity function. All test statistics fall inside the critical intervals. It does not contradict the NHPP assumption. On Figure 8 is shown the step function of the failure numbers and corresponding estimated intensity function. Estimated parameter β is equal to 2.

Let us generate the following sequence of time intervals:

$$t_1 = \sqrt{2},\ t_2 = \sqrt{4},\ ...\ ,\ t_n = \sqrt{2n}\ .$$

Intervals between events are

$$\delta_1^2 = \sqrt{2},\ \delta_2^2 = \sqrt{4} - \sqrt{2},\ ...\ ,\ \delta_n^2 = \sqrt{2n} - \sqrt{2(n-1)}\ .$$

In transformed time

$$\Delta_k = \int_0^{\sqrt{2k}} \upsilon d\upsilon = \frac{\upsilon^2}{2}\Big|_0^{\sqrt{2k}} = \frac{2k}{2} = k$$

and inter events intervals in transformed time exactly equals to 1. Check this sequence of time intervals using our procedure with Power intensity function. Estimated parameter β is equal to 2. Seven test statistics fall outside the critical intervals. It contradicts the NHPP assumption. On Figure 9 is shown the step function of the failure numbers and corresponding estimated intensity function.

Now let us mix two sequences of these time intervals.

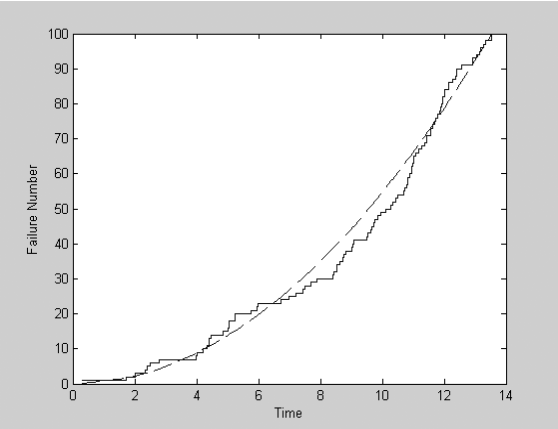

Figure 8. *The Plot of the generated NHPP step function of the failure numbers and corresponding estimated intensity function against time*

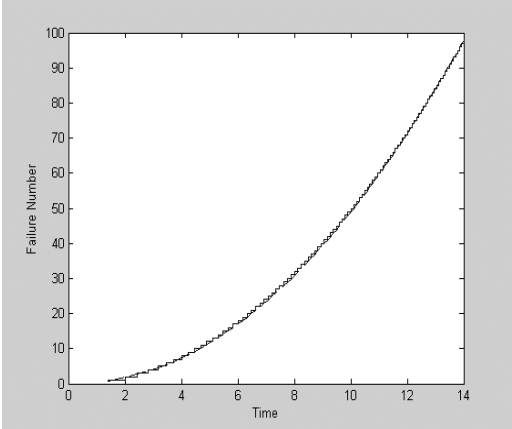

Figure 9*. The Plot of the step function of the failure numbers and corresponding estimated intensity function against time*

$$\delta_1^* = (1-\alpha)\delta_1^1 + \alpha\delta_1^2$$

.....

$$\delta_k^* = (1-\alpha)\delta_k^1 + \alpha\delta_k^2$$

.....

$$\delta_n^* = (1-\alpha)\delta_n^1 + \alpha\delta_n^2$$

If $\alpha = 0$, the sequence $\delta_1^*, \delta_2^*, \ldots, \delta_n^*$ is NHPP. If $\alpha = 1$, the sequence $\delta_1^*, \delta_2^*, \ldots, \delta_n^*$ is non-homogeneous deterministic process. Changing α from 0 to 1, we receive the set of perturbed processes from NHPP to deterministic process. Using procedure of power testing we received results shown on Figure 10.

The sequence of purely exponential random variables is recognized by our tests as a HPP. Slightly perturbed sequence, e.g. α = 0.1 ÷ 0.2, of exponential random variables are recognized by our tests as being HPP. Other processes are recognized by our tests as not being HPP.

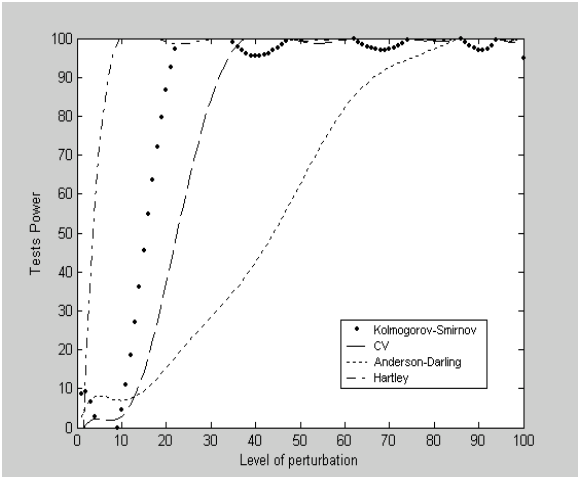

Figure 10. *Comparison of Tests Power with Power Function Recognition*

4. Conclusion

The test method proposed in this article is based on time transformation of the given process and the estimation of intensity function using the Maximum Likelihood Method. Critical values of the statistics were obtained by a repeated sampling of NHPP with estimated intensity function. Computer-intensive procedure allows us to suppose that the given sample path while testing the hypothesis belongs to NHPP.

We checked the procedure on different reliability data sets. We used failure data existing in literature and compared our conclusions to authors' results. In most cases there was the coincidence of results. Also we presented our own numerical example and showed how the above described procedure functions. Our procedure was verified on a set of the stochastic processes with known probabilistic structures, which includes HPP and a deterministic renewal process. It was demonstrated how our procedure identifies the underlying process.

The main weakness of our method is the fact that we worked only with NHPP having Log-linear and Power-law intensity functions, because there are efficient and easy to apply ML Method for estimation of the parameters. It would be desirable the develop numerical efficient methods to fitting the empirical cumulative failure rate to a wider class of functions, together with the corresponding methods of parameter estimation.

The above numerical results demonstrate the performance of our method to test the Nonhomogeneous Poisson Process hypothesis versus an alternative belong to a set of random processes obtained as addition of a "noise" to a pure NHPP. The best of our knowledge, there are no experimental data on similar test performance. As soon as they will become, we will compare the performance of our procedure versus the alternative.

Acknowledgment

This research was supported by research grant from the Negev Academic College of Engineering (NHPP), Beer-Sheva, Israel.

Reference

1. Akaike, H. (1974), A new Look at the Statistical Model Identification, IEEE Transactions on Automatic Control, AC-19(6), pp.716-723.
2. Anderson, T. W. and Darling, D. A. (1952), Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Process, The Annals of Mathematical Statistics, 23(2), pp. 193-212.
3. Anderson, T. W. and Darling, D. A. (1954), A Test of Goodness of Fit, Journal of the American Statistical Association, 49(268), pp. 765-769.
4. Ansel, J., Archibsald, T., Dagpunar, J and Thomas, L. (2001), Estimating Asset Life of Major Equipment, Proceeding of ESREL2001.
5. Ansell, J. B. and Phillips, M. J. (1989), Practical problems in the Statistical Analysis of Reliability Data, Applied Statistics, 38(2), pp. 205-247.
6. Arkin, B. L. and Leemis, L. M. (2000), Nonparametric Estimation of the Cumulative Intensity Function for a Nonhomogeneous Poisson Process from Overlapping Realization, Management Science, 46(7), pp. 989-998.
7. Ascher, H. E. and Feingold, H. (1984), Repairable Systems Reliability. Marcel Dekker, New York.
8. Ascher, H. E. and Hansen, C. K., (1998), Spurious Exponentiality Observed When Incorrectly Fitting a Distribution to Nonstationary Data. IEEE Transactions on Reliability, 47(4), pp. 451-459.
9. Bain, L. J., Engelhardt, M. And Wright, F. T. (1985), Testing for an Increasing Trend in the Intensity of a Poisson Process: a Power Study, Journal of the American Statistical Association, 80. pp. 419-422.
10. Baker, R. D. (1996), Some new tests of the power-law process, Technometrics, 38, pp. 256-265.

11. Bassin, W. M. (1973), A Bayesian Optimal Overhaul Interval Model for the Weibul Restoration Process Case, Journal of the American Statistical Association, 68(343), pp. 575-578.
12. Bergman, B. and Klefsjö, B. (1984), The Total Time to Test Concept and Its Use in Reliability Theory, Oper. Research, 32(3), pp. 596-606.
13. Bhattacharjee, M. (2001), Inference for Non-Homogeneous Poisson Processes: Models for Repairable Systems, ISI Session, Seul, August, 2001, pp. 1-16.
14. Black, S. E. and Rigdon, S. E. (1996), Statistical Inference for a Modulated Power Law Process, Journal of Quality Technology, 28(1), pp. 81-90.
15. Bracquemond, C., Chevalier, M. and Gaudoin, O. (2001), Modeling the Aging of Non Repairable Systems in Discrete Time, Proceedings of ESREL2001.
16. Çinlar, E. (1975), *Introduction to Stochastic Processes*, Prentice-Hall, New-Jersey.
17. Coetzee, J. L. (1997), The Role of NHPP Models in the Practical Analysis of Maintenance Failure Data, Reliability Engineering and System Safety, 56, pp. 161-168.
18. Cox, D. R. and Lewis, P. A. W. (1966), *The Statistical Analysis of Series of Events,* Chapman and Hall, London.
19. Crow, L.H. (1974), Reliability Analysis for Complex, Repairable Systems, Reliability and Biometry, pp. 379-410.
20. Crow, L.H. (1990), Evaluating the Reliability of Repairable Systems, Proceedings of Annual Reliability and Maintainability Symposium, pp. 275-279.
21. Crowder, M. J., Kimber, A. C., Smith, R. L. and Sweeting, T. J. (1991), *Statistical Analysis of Reliability Data*, Chapman and Hall, New York.
22. Dohi, T., Takeita, K. and Osaki, S. (2000), Graphical Methods for Determining/Estimating Optimal Repair-Limit Replacement Systems, International Journal of Reliability, Quality and Safety Engineering, 7(1), pp. 43-60.
23. Epstein, B. (1960), Tests for the Validity of the Assumption that the Underlying Distribution of Life is Exponential. Part I, Technometrics, 2(1), pp. 83-101.
24. Epstein, B. (1960), Tests for the Validity of the Assumption that the Underlying Distribution of Life is Exponential. Part II, Technometrics, 2(2), pp. 167-183.
25. Frenkel, I. B., Gertsbakh, I. B. and Khvatskin, L. V. (2003), Parameter Estimation and Hypotheses Testing for Nonhomogeneous Poisson Process, Transport and Telecommunication, 4(2), pp. 9-17.
26. Gaudoin, O. (1992), Optimal Properties of the Laplace Trend Test for Software-Reliability Models, IEEE Transactions on Reliability, 41(4), pp. 525-532.
27. Gaudoin, O., Yang, B. and Xie, Min. (2003), A Simple Goodness-of-Fit Test for the Power-Law Process, Based on the Duane Plot, IEEE Transactions on Reliability, 52(1), pp. 69-74.
28. Gertsbakh, I. B. (2000), *Reliability Theory with Applications to Preventive Maintenance*. Springer Verlag.
29. Gnedenko, B. V, Belyaev, Yu. K. and Solovyev, A. D. (1969), Mathematical Methods in Reliability Theory. Academic Press, New York.
30. Gokhale, S. S. and Trivedi, K. S. (1996), Log-Logistic Software Reliability Growth Model, Proceedings of ISSRE1996.
31. Guess, F., Hollander, M. and Proschan, F. (1986), Testing Exponentiality Versus a Trend Change in Mean Residual Life, The Annals of Statistics, 14(4),pp. 1388-1398.
32. Hartley, H. O. (1950), The maximum F-ratio as a short-cut test for heterogeneity of variance, Biometrica, 37, pp.308-312.
33. Jani, P. N., Shanubhogue, A. and Muralidharan, K. (1997), Some Tests for a Poisson Process and Their Power with Respect to Trend in the Intensity, Commun. Statist. – Theory Meth., 26(11), pp. 2671-2688.
34. Kaminsky, F. C. and Rumpf, D. L. (1977), Simulating Nonstationary Poisson Processes: a Comparison of Alternatives Including the Correct Approach, Simulation, 7, pp. 17-20.
35. Kao, E. P. C. and Chang, S.-L. (1988), Modeling Time-Dependent Arrivals to Service Systems, Management Science, 34(11), pp. 1369-1379.
36. Klefsjö, B. and Kumar, U. (1992), Goodness-of-fit Tests for the Power-Law Process Based on the TTT-Plot, IEEE Transactions on Reliability, 41(4), pp. 593-598.

37. Klein, R. W. and Roberts, S. D. (1984), A Time-varying Poisson Arrival Process Generator, Simulation, 10, pp. 193-195.
38. Kuhl, M. E. and Bhairgond, P. S. (2000), Nonparametric Estimation of Nonhomogeneous Poisson Processes using Wavelets, Proceedings of the 2000 Winter Simulation Conference, pp. 562-571.
39. Kuhl, M. E., Damerdji, H. and Wilson, J. R. (1998), Least Square Estimation of Nonhomogeneous Poisson Processes, Proceedings of the 1998 Winter Simulation Conference, pp. 637-645.
40. Kuhl, M. E. and Lim, S. E. (1999), Sensitivity Analysis of Simulation Output to Parameters of Nonhomogeneous Poisson Processes, Proceedings of the 1999 Winter Simulation Conference, pp. 462-465.
41. Kuhl, M. E., Wilson, J. R. and Jonson, M. A. (1997), Estimating and Simulating Poisson Processes Having Trends or Multiple Periodicities, IIE Transactions, 29, pp. 201-211.
42. Kuhl, M. E. and Wilson, J. R. (2001), Modeling and Simulating Poisson Processes Having Trends or Nontrigonometric cyclic effects, European Journal of Operational Research, 133, pp. 566-582.
43. Kvaløy, J. T. and Lindqvist, B. H. (1998), TTT-Based Tests for Trend in Repairable System Data, Reliability Engineering and System Safety, 60, pp. 13-28.
44. Kvaløy, J. T., Lindqvist, B. H. and Malmedal, H. (2001), A Statistical Test for Monotonic and Non-Monotonic trend in Repairable Systems, Proceedings of ESREL2001.
45. Lawless, J. F. and Thiagarajah, K. (1996), A point-process model incorporating renewals and time trend, with application to repairable systems, Technometrics, 38, pp. 131-138.
46. Lee, S., Wilson, J. R. and Crawford, M. M. (1001), Modeling and Simulation of a Nonhomogeneous Poisson Process having Cyclic Behavior, Communication of Statistics – Simulation, 20(2&3), pp. 777-809.
47. Leemis, L. W. (1988), Input Modeling, Proceedings of the 1998 Winter Simulatiom Conference, pp. 15-22.
48. Leemis, L. W. (1991), Nonparametric Estimation of the Cumulative Intensity Function for Nonhomogeneous Poisson Process, Management Science, 37(7), pp. 886-900.
49. Lewis, P. A. W. and Robinson, D. W. (1974), Testing for a Monotone Trend in a Modulated Renewal Process, Reliability and Biometry, pp. 163-182.
50. Lewis, P. A. and Shelder, G. S. (1976), Statistical Analysis of Non-stationary Series of Events in a Data Base Systems, IBM J. Res. Development, pp. 465-482.
51. Lewis, P. A. and Shelder, G. S. (1976), Simulation of Nonhomogeneous Poisson Process with Log Linear Rate Function, Biometrica, 63(3), pp. 501-505.
52. Lewis, P. A. and Shelder, G. S. (1979), Simulation of Nonhomogeneous Poisson Process by Thinning, Naval Research Logistics Quarterly, 26, pp. 403-413.
53. Lilliefors, H. W. (1969), On the Kolmogorov-Smirnov Test for the Exponential Distribution with Mean Unknown, Journal of the American Statistical Association, 64, pp. 387—389..
54. Lindqvist, B. H. (1997), Statistical Modeling and Analysis of Repairable Systems, Proceedings of the 1st International Conference on Mathematical Methods in Reliability, Bucharest, Romania, September, 16-19, 1997, (MMR'97).
55. Lomnicki, Z. A. (1966), A Note on the Weibull Renewal Process, Biometrica, 53(3 and 4), pp. 375-381.
56. López-Swgovia, L., Villaseñor-Alva, J. A. and Vaquera-Huerta, H. (2002), Two Goodness of Fit Tests for Non Homogeneous Poisson Processes, Agrociencia, 36(3), pp. 703-712.
57. MacLean, C. J. (1974), Estimation and Testing of an Exponential Polinomial Rate Function with the Nonstationary Poisson Process, Biometrica, 61(1), pp. 81-85.
58. Mann, H. B. (1945), Nonparametric Tests Against Trend, Econometrica, 13(3), pp. 245-259.
59. Massey, W. A., Parker, G. A. and Whirtt, W. Estimating the Parameters of a Nonhomogeneous Poisson Process with Linear Rate, Telecommunication Systems, 5, pp. 361-388.
60. Meeker, W. D. and Escobar, L. A. (1998), *Statistical Methods for Reliability Data,* Wiley and Sons.

61. MIL-HDBK-189 (1981), Reliability Growth Management. HQ, US Army Communications Research and Development Command. ATTN: DRDCORT, Fort Monmouth, New Jersey.
62. Moller, S. K. (1976), the Rasch-Weibull Process, Scandinavian Journal of Statistics, 3, pp. 107-115.
63. Muralidharan, K., (2001), On testing of Parameters in Modulated Power Law Process, Applied Stochastic Models in Business and Industry, 17, pp. 331-343.
64. Nechval, N. A., Vasermanis, E. K., Rozevskis, U., Rozite, K. and Nechval, K. N. (2003), Characterization Theorems and Goodness-of-Fit Testing on Smal Data Samples, Transport and Telecommunication, 4(2), pp. 92-100.
65. Park, W. J. and Kim, Y. G., (1992), Goodness-of-Fit Tests for the Power-Law Process, IEEE Transactions on Reliability, 41(1), pp. 107-111.
66. Park, W. J. and Seon, M., (1994), More Goodness-of-Fit Tests for the Power-Law Process, IEEE Transactions on Reliability, 43, pp. 275-278.
67. Proschan, F. (1963), Theoretical Explanation of Observed Decreasing Failure Rate, Technometrics, 5(3), pp. 375-383.
68. Pulcini, G. (2001), A Bounded Intensity Process for the Reliability of Repairable Equipment, Journal of Quality Technology, 33(4), pp. 480-492.
69. Rigdon, S. E. (1989), Testing Goodness-of-Fit for the Power Law Process, Commun. Statist. – Theory Meth., 18(12), pp. 4665-4676.
70. Rigdon, S. E. and Basu, A. P. (1989), The Power Law Process: A Model for the Reliability of Repairable Systems, Journal of Quality Technology, 21(4), pp.251-260.
71. Rigdon, S. E. and Basu, A. P. (1990), The Effect of Assuming a Homogeneous Poisson Process When the True Process is a Power Law Process, Journal of Quality Technology, 21(4), pp.111-117.
72. Saldanha, P. C., de Simone, E., A. and Frutuoso e Melo, P., F. (2001), An Application of Non-Homogeneous Poisson Processes to the Reliability Analysis of Service Water Pump, Nuclear Engineering and Design, 210, pp.125-133.
73. Schmee, J., Gladstein, D. and Nelson, W. (1985), Confidence Limits for Parameters of a Normal Distribution from Singly Censored Samples, Using Maximum Likelihood, Technometrics, 27(2), pp. 119-128.
74. Serfling, R. J. (1974), Research in Point Process, with Applications in Reliability and Biometry, Reliability and Biometry, pp. 109-127.
75. Shin, I., Lim, T. J. and Lie, C. H., (1996), Estimation Parameters of Intensity Function and maintenance Effect for Repairable Unit. Reliability Engineering and System Safety, 54, pp. 1-10.
76. Schneidewind, N. F. (1993), Software Reliability Model with Optimal Selection of Failure Data, IEEE Transactions on Software Engineering, 19(11), pp. 1095-1104.
77. Thompson, W. A. Jr. (1981), On the Foundations of Reliability, Technometrics, 23(1), pp. 1-13.
78. Villaseñor-Alva, J. A., and Díaz-Carreño, M. A. (2003), Non-Parametric Tests for Non-Homogeneous Poisson Processes, Agrociencia, 37(1), pp. 21-31.
79. Wang, K. S., Po, H. J., Hsu, F. S. and Lui, C. S. (2002), Analysis of Equivalent Dynamic Reliability With Repairs Under Partial Information, Reliability Engineering and System Safety, 76, pp. 29-42.
80. Weckman, G. R., Shell, R. L. and Marvel, J. H. (2001), Modeling the Reliability of Repairable Systems in the Aviation Industry, Computers and Industrial Engineering, 40, pp. 51-63.
81. Xie, M., Yang, B. and Gaudoin, O. (2001), Regression Goodness-of-Fit Test for Software Reliability Model Validation, Proceedings of ISSRE2001.
82. Yañez, M., Joglar, F. and Nodarres, M. (2002), Generalized Renewal Process for Analysis of Repairable Systems with limited Failure Experience. Reliability Engineering and System Safety, 77, pp. 167-180.
83. Zeljkoviæ, V. and Begenišiæ, N. (2001), Software Reliability – Different Model Analyses, Proceeding of TELFOR2001, Telecommunication Forum, November 20-22, 2001, Beograd.
84. Zhao, M. and Xie, M. (1992), On the Log-Power NHPP Software Reliability Model, Proceedings of the 3^{d} Int. Symp. On Software Reliability Engineering, North Carolina, USA, IEEE Comp.-Society Press, 7-10 Oct. 1992, North Carolina, USA, pp. 14-22.

COMBINATORIAL AND PROBABILISTIC PROPERTIES OF LOMONOSOV's "TURNIP"

I. Gertsbakh[1], Y. Shpungin[2]

[1] *Ben Gurion University of the Negev, Beersheva, P.O.Box 653, Israel*
e-mail: eliahu@cs.bgu.ac.il

[2] *Negev Academic College of Engineering, Beersheva, 84100, Israel*
e-mail: yosefs@nace.ac.il

Abstract

This paper is a review of a very efficient combinatorial approach for Monte Carlo (MC) estimation of networks reliability parameters. It is well known that the main problem of the Crude Monte Carlo and various versions of it is unbounded growth of the relative error. This new and very promising approach was first suggested by M. Lomonosov in 1974 [1] and developed later in a series of works [2,3,4,5,6]. The method eliminates the rare event problem, has several computational advantages and is very useful for calculating different reliability parameters such as the stationary network availability and so-called stationary down-up transition rate.

Key words: terminal connectivity; network reliability; Monte Carlo; closure; trajectory; transition rate

1. Introduction

Networks and their reliability is a subject of great interest. Many researches in various directions take place in this field. This paper presents one of the most useful approaches for estimating reliability parameters of static and dynamic networks. The principal idea of this approach was first suggested in [1].

1.1. What is a Network?

The common feature for all networks is that they consist of *vertices* and *edges*. There exist many types of networks varying in their performance definitions and therefore with different concepts of reliability. For our purposes, we define a network in the following manner:

By network $N=(V, E, T)$ we denote an undirected graph with node-set V (vertices) , edge-set E and set T of special nodes named *terminals* (see Fig. 1.a). T is a subset of V.

We define nodes as *reliable* and edges as *not reliable*. In other words, nodes never fail and edge $e \in E$ fails (is being erased) with probability $q(e)$. A *state* of a network is a set of *non-erased* edges. By the definition, the network is in UP state if all its terminals are connected, i.e. if between each pair of terminals there exists a path that connects them. Otherwise we say that a network is in the DOWN state. Two most popular cases of the operational criteria for networks are the s-t connectivity and all-terminal connectivity. Fig. 1.a presents a network N with two terminals: s and t. In Fig. 1.b we see a DOWN state of N: {2,3,5,7}. Fig. 1.c shows an UP state of N: {4,7}.

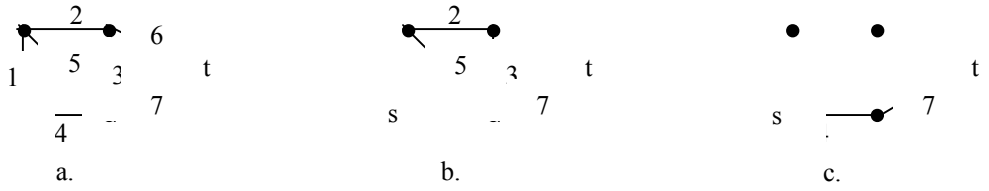

Figure 1. *The s-t terminal Network and its DOWN and UP states. Bold points are the vertices. Terminals are denoted by letters s,t.*

The common static network problem is the calculation of its *Reliability*, i.e. the probability that a network is in the UP state. Note that the straightforward calculation of reliability is very time consuming, even for moderate size networks. Also the well known path-set and/or cut-set methods demand computation time which grow exponentially with the network size. Computational difficulties in the direct calculation of static network reliability stimulated great interest in developing Monte Carlo methods.

We do not intend to review all the approaches in this direction. The greatest part of these works is related to so-called Crude Monte Carlo (CMC) or its modifications. The main deficiency of these methods is that they are very inefficient in two extreme cases: highly reliable and highly unreliable networks (the so called rare event phenomenon). Note that the highly reliable networks have great interest in many important applications.

In this paper we present a very efficient MC approach suggested in [1] and developed in [2,3,4,5,6]. This method is especially useful for the extreme cases mentioned above. It *in principle* eliminates the rare event phenomenon and has, in addition, some useful properties for its applications in static and dynamic networks. (If edge *e* has lifetime $\xi(e),\ e \in E,$ we call such network *dynamic*).

1.2. Reliability and Crude Monte Carlo

Let us begin with a very simple network shown on Fig. 2. Suppose that the operational criterion is all-terminal connectivity and the edge probabilities of being up are equal: $p(1)=p(2)=p$.

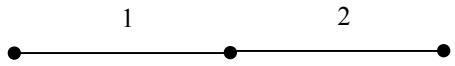

***Figure 2**. A series network*

Clearly that the only UP state here is the state $S=\{1,2\}$ and the reliability of the network $R(N) = p^2$. Suppose now that we want to estimate this reliability using CMC. Denote by X the random variable which characterizes the state of the network:

$$X = \begin{cases} 1, & \text{if the state is UP} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $p(X=1) = p^2$ and $E[X] = p^2$. The procedure of CMC is the following

(i) put $\hat{R} := 0$;

(ii) simulate the states of edges 1 and 2 with the given probability *p* and determine *X*;

(iii) $\hat{R} := \hat{R} + X$;

(iv) repeat the above operations M times;

(v) the estimate of the *R* is $\hat{R} := \hat{R}$/M.

Clearly that $\hat{R}$ is an unbiased estimator of *R*.

The deficiency of this procedure lies in the simulation of edges' states. If the edges are highly reliable, say $p \to 1$, we will almost never observe the down states of the edges. This means that the estimated value of *R* may be 1! Obviously, this result is not satisfactory.

Let us look at this situation more formally. One of the measures of the MC methods is the relative error, which in our case equals

$$\delta_{CMC} = \frac{\sqrt{Var(1-\hat{R})}}{1-R} = \sqrt{\frac{Var(X)}{(1-p^2)^2 \cdot M}} = \sqrt{\frac{p^2}{1-p^2}} \cdot \frac{1}{\sqrt{M}}. \quad (1)$$

From this formula we see that when $p \to 1$, then for fixed M the value of the relative error tends to *infinity*, and this is the main obstacle for using CMC.

1.3. The CMC as an "*Urn*" Scheme

To explain the advantages of the Lomonosov's idea, let us look at the CMC from a more general point of view. Consider an *urn U* with a large number of balls *b* in it. Suppose that each ball *b* is marked with some value *z*(*b*), and we want to calculate the sum of *z*(*b*) over *b* in *U*:

$$Z = \sum_{b \in U} z(b). \quad (2)$$

This completely corresponds to computation of network reliability. For this case, the balls *b* are the states, and *z*(*b*) are defined as 0 for any DOWN state and as the probability of the state if the state belongs to the set UP. Therefore, *Z* becomes the reliability of the network. Because of a very large number of balls in *U*, we usually cannot compute exactly the whole sum, and we are forced to estimate *Z*. To convert the expression in (2) into MC scheme, we introduce the probability distribution *p(b)* on *U*. Ball *b* will be drawn with probability *p*(*b*). Then one may express the sum *Z* in the following form:

$$Z = \sum_{b \in U} p(b) \cdot \frac{z(b)}{p(b)} = E[Y(b)], \quad (3)$$

where $Y(b) = \frac{z(b)}{p(b)}$. (Note that values of *Y*(*b*) are 0 or 1 for the states DOWN and UP, respectively). The main idea of this representation is that the sum *Z* is expressed as an *expectation* of some random variable.

Now the CMC scheme for evaluating $Z = E[Y]$ can be stated as follows:

- draw balls b_i M times from the urn U, with probability $p(b_i)$;
- $$\hat{Y} = \frac{\sum_{i=1}^{M} Y(b_i)}{M}$$
- calculate, which is an *unbiased* estimator of *Z*.

As in the example in (1.2), it is easy to get the general form for the relative error:

$$\delta_{CMC} = \sqrt{\frac{R}{1-R}} \cdot \frac{1}{\sqrt{M}}. \quad (4)$$

Again we see the unbounded growth of the relative error for highly reliable networks. Let us emphasize once more that it is the main deficiency of the CMC. The reason for this lies in the above described urn scheme. Namely, each ball, which represents the state, is drawn from the urn with the probability of that state. Therefore the growth of the relative error is caused by a bad definition of probability measure *p*(*b*) on network states.

Various improvements of CMC have been suggested in order to reduce the unbounded growth of the δ_{CMC}. The approach suggested in [2] is referred to in this paper as Lomonosov's turnip.

This is a *different urn* scheme, which eliminates the rare event phenomenon and is very efficient for computing the reliability of highly reliable networks.

2. The Idea of the Turnip

The turnip is constructed using three basic ideas. The first one is introducing some *artificial* random process associated with each edge. The second one is defining the *balls b* in the *urn* scheme as certain *trajectories* which do not depend on the probabilities of the network states. The third one is using a special combinatorial operation (*closure*) which considerably accelerates the computation procedure.

2.1. Parameterization and *Artificial* Creation Process

Introduce for each edge an *artificial* creation process (see Fig. 3). Initially, at t=0, every edge e is down. Edge is "born" independently of others and remains *up* forever. The random moment $\xi(e)$ of edge "birth" is exponentially distributed:

$$P\{\xi(e) \le t\} = 1 - \exp(-\lambda(e) \cdot t), \quad e \in E. \tag{5}$$

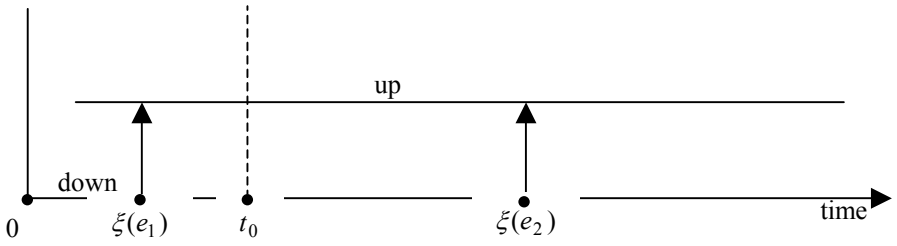

Figure 3. *Independent appearance of edges in time. At the instant t_0, edge e_1 does exist while e_2 does not*

Fix now an *arbitrary* moment t_0 (see Fig. 3), in particular, $t_0 = 1$. Chose for each edge e, the "birth" rate $\lambda(e)$ so that the following condition holds:

$$P\{\xi(e) > t_0\} = \exp(-\lambda(e) \cdot t_0) = q(e). \tag{6}$$

This means that in this *dynamic* process, the probability of being *down* at t_0 for each edge *coincides* with the *static down*-probability $q(e)$. At t=0 our network is DOWN. At some random moment $\xi(N)$ the network becomes UP (and remains in the UP forever). Let us denote by $P\{\xi(N) \le t_0\}$ the probability that at moment t_0 in the creation process the whole network N is UP. Then we have:

$$R(N) = P\{\xi(N) \le t_0\}. \tag{7}$$

In words: the static probability $R(N)$ that the network is UP coincides with the probability that in the creation process, the state becomes UP before t_0.

2.2. Closure

One of the important advantages of our approach is the essential use of *combinatorial* features of the network, namely the operation of *closure*. We give an intuitive explanation for closure. For more precise definition see [2].

Suppose that the edges in the creation process appear randomly and independently. In Fig. 4.a a network with four nodes and five edges is given. We are going to use this network in the following examples for two operational criteria: all terminal connectivity and *s-t* connectivity.

Suppose that first edges 1 and 2 are born. At this moment the nodes associated with edge 3 are already connected by a *path* formed by edges 1 and 2. Therefore edge 3 does not affect the connectivity of the component {1, 2, 3} and becomes *irrelevant* to the further evolution of the network.

For each state $S \subseteq E$ of the network we define a *closure* as a subset of edges in *E* so that it equals to the *union* of *S* and all irrelevant edges (for given operational criterion). For example, in Fig. 4.b the set $\sigma_{21} = \{1,2,3\}$ is a closure for each of the following states: {1,2}, {1,3}, {2,3}. Let us carry out the closure operation within each connected component of the network. Then we call the collection of all such components a *super-state.* (A single node is also a closed component).

2.3. Turnip as Evolution Process with Closure

On Fig. 4.b we present the turnip as the creation process defined in 2.1 together with the closure operation defined in 2.2. As an example, we consider a network *N* with four nodes and five edges. The operational criterion is the all-terminal connectivity.

Suppose that initially all edges are in the *DOWN* state, i.e. the initial *super-state* is a collection of one-node components. The zero-level of the turnip is the set *V* without any edge (no edge was born). The first level shows all possible evolution results from the zero level which appear as a result of a birth of a single edge. There are 5 such super-states. The second level of the turnip shows what happens when a second edge is born. We distinguish 6 such super-states $\sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}$. It is important to stress that these super-states are shown together with the relevant closure. For example, suppose that after edge 1, edge 3 is born. Then edge 2 can be added to the existing edges, and the corresponding super-state is σ_{21}. Similarly, the same σ_{21} appears if 2 is born after 1.

The last level consists of one super-state containing one component. Note that in the case of all-terminal connectivity, only at this level the corresponding state is the UP-state. In the case of the *s-t* connectivity the UP states may be on each level. For example, consider the network on Fig. 4.c, which has the terminals *s* and *t*. We see that the UP states are situated on levels 2 and 3 (The corresponding super-states on Fig. 4.b are double-circled).

An important feature of the turnip is that simulating the transitions from one super-state to another is very easy. Compute for example the probability $P(\sigma_{11} \to \sigma_{21})$ of the transition from σ_{11} to σ_{21}. Suppose that the birth rates for edge *k* is $\lambda(k)$, $k = 1,...,5$. Then, by the well known fact, we have:

$$P(\sigma_{11} \to \sigma_{21}) = \frac{\lambda(2)+\lambda(3)}{\lambda(2)+\lambda(3)+\lambda(4)+\lambda(5)} = \frac{\lambda(\sigma_{11})-\lambda(\sigma_{21})}{\lambda(\sigma_{11})}, \qquad (8)$$

where $\lambda(\sigma_{ij})$ denotes the total birth rate for the super-state σ_{ij}. Indeed, the transition $\sigma_{11} \to \sigma_{21}$ takes place if and only if edge 2 or 3 is born out of four possibilities of the birth of edges 2, 3, 4 and 5.

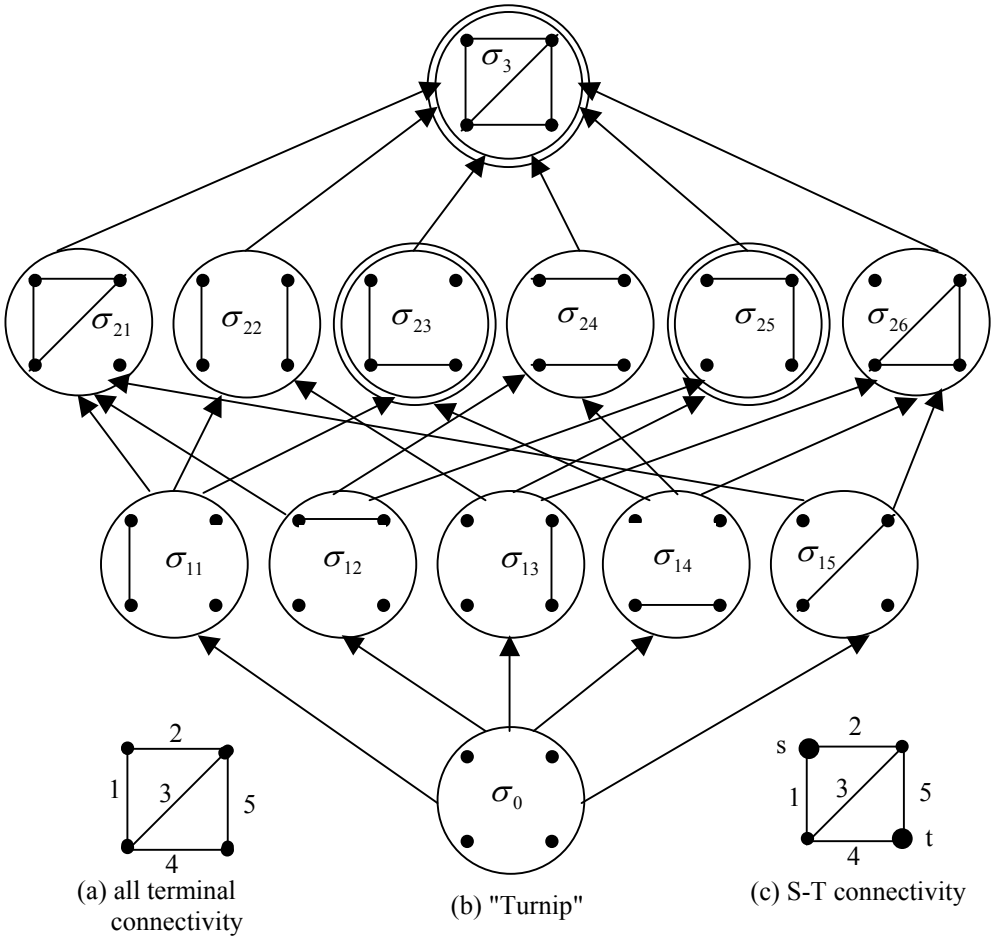

Figure 4. *The "Turnip" for two cases of connectivity*

Actually, the "turnip" describes an artificial creation process with closure, and the probability that the network in this process will be UP at some moment t_0, coincides with the corresponding static probability of the network. For more precise description of the process see [2].

Consider a random process $\sigma(t)$ whose states are the super-states of the above described network evolution process. (For example $\sigma(t=0)=\sigma_0$). We already mentioned that each state corresponds to some super-state. The following claim was proved in [2].

Claim 1. (i) $\sigma(t)$ is a Markov process;

(ii) The time spent by $\sigma(t)$ in a particular super-state σ' is distributed as $\exp(\lambda(\sigma'))$;

(iii) Let σ'' be the direct successor of σ'. Then the transition $\sigma' \to \sigma''$ takes place with the probability $P(\sigma' \to \sigma'') = (\lambda(\sigma') - \lambda(\sigma''))/\lambda(\sigma')$.

Remark. Each successor σ'' of σ' is obtained by merging exactly two components of σ'. The process $\sigma(t)$ is called in [2] the Merging Process (MP).

Define now a new term, *trajectory*, which plays the central role in the Merging Process (and therefore also in the corresponding Monte Carlo scheme). A trajectory is a sequence $u=(\sigma_0,\sigma_1,...,\sigma_r)$ of super-states such that σ_0 is the trivial initial super-state, each σ_i is the direct successor of σ_{i-1}, and r is the first i such that σ_i is UP. For example, the sequence $(\sigma_0,\sigma_{11},\sigma_{23},\sigma_3)$ on the Fig. 4.b is a trajectory.

Now, in terms of the trajectories, the network is UP at moment t if there exists at least one trajectory which reaches the UP state before t. It is easy to calculate the probability $p(u)$ for a trajectory $u = (\sigma_0, ..., \sigma_r)$:

$$p(u) = \prod_{i=0}^{r-1} P(\sigma_i \to \sigma_{i+1}). \tag{9}$$

Suppose that the evolution goes along the trajectory u. Denote by $P(t|u)$ the probability that the UP-state (i.e., σ_r) will be reached before time t *given* that it goes along u. By property (ii) of the claim 1, our MP is sitting in each super-state σ_j an exponentially distributed random time $\tau(\sigma_j)$ and due to the Markovian property, the total evolution time along the trajectory u is a sum of the respective exponential random variables. More formally,

$$P(t \mid u) = P(\tau(\sigma_0) + \tau(\sigma_1) + ... + \tau(\sigma_{r-1}) < t \mid u = (\sigma_0, ..., \sigma_r)). \tag{10}$$

Note that given u, this quantity can be computed directly as a convolution of respective exponents.

Now, the probability that at moment t the MP process reaches UP equals

$$R(N) = P\{\xi(N) \le t\} = \sum_{u \in U} p(u) P(t \mid u), \tag{11}$$

where U is the set of all trajectories. The latter expression has the form of an expectation as in (3) and determines in fact, the simulation strategy for its evaluation:

(i) put $\hat{R} := 0$;

(ii) *generate* trajectory u from the trivial super-state to the super-state in the UP, with the probability (9);

(iii) *calculate* $\hat{R} := \hat{R} + P(t \mid u)$;

(iv) repeat the (ii) and (iii) M times;

Then the $\frac{\hat{R}}{M}$ is an unbiased estimator of $R(N)$.

In terms of the *urn scheme* (see 1.3) the trajectory is a *ball* and it is drawn with probability (9) which *does not depend* on the edge probabilities. This, at least intuitively, indicates that in the MP scheme the rare event phenomenon does not exist. The following claim was proved in [3].

Claim 2. For a given number of nodes and a given operational criterion the coefficient of variation δ^2_{MP} is bounded uniformly for all $t \in [0, \infty)$ and all λ-vectors satisfying $\max \lambda(\cdot) / \min \lambda(\cdot) \le C$ (for any C).

Claim 2 assures (through the Tshebyshev inequality), that for any given number of nodes n and rates $\lambda(e)$, and for any positive ε, δ, there exists a sample size N such that for all $0 \le t < \infty$, we have

$$1 - \varepsilon < \frac{\hat{R}(N)}{R(N)} < 1 + \varepsilon \tag{12}$$

with probability at least $1 - \delta$. Note that this property does not hold for the CMC.

3. Applications of Turnip

3.1. Availability $A_N(t)$

The static reliability *R(N)* also expresses the equilibrium instant availability of *N*. Suppose that the life of each edge *e* is described by an alternating renewal process. Assume that for this process there exist mean *up* and *down* periods. Consider the process *E(t)*- the set of all edges from *E* which are UP at moment *t.* Then the availability $A_N(t)$ as $t \to \infty$ of the network *N* equals the static probability *R*(*N*), with edge *down*-probability given by $q(e) = \frac{\lambda(e)}{\lambda(e) + \mu(e)}$, where $\lambda(e)$ is the failure rate and $\mu(e)$ is the repair rate. For more details see Gnedenko et al [7].

3.2. The Mean Period T_{UP} and the Transition Rate $\Phi(N)$

One of important characteristics of the *dynamic* network is the mean equilibrium *up* and *down* periods of the network, T_{UP} *and* T_{DOWN} respectively. Let us define T_{UP} (T_{DOWN} is similar). Consider the sequence $\{U_i\}, i = 1,...,k$ of *UP* periods. Then, it may be proved that, as $k \to \infty$, the sequence $\{U_k\}$ converges in probability to limit U with $E[U] = \lim_{k\to\infty} E[U_k]$. Denote $T_{UP} = E[U]$ (see [7]). Computing the values T_{UP} and T_{DOWN} is a very difficult task. One can compute them using the MP. At first, note that the following formulas are valid ([7]):

$$R(N) = \frac{T_{UP}}{T_{UP} + T_{DOWN}}, \tag{13}$$

$$\Phi(N) = \frac{1}{T_{UP} + T_{DOWN}}, \tag{14}$$

where $\Phi(N)$ is the so-called stationary transition rate (see its definition below). Clearly, computing $\Phi(N)$ opens the way for computing T_{UP}. The turnip effectively helps in doing so.

Let us now describe the notion of $\Phi(N)$, which is itself an interesting characteristic of the renewable network. By definition, $\Phi(N)$ is the transition rate from DOWN to UP. In fact, a transition DOWN $\to$ UP may take place only from so-called *border* states. A border state is such particular DOWN state of the network which may be transformed in an UP state by adding a single edge. Denote *BD* the set of all border states. For any border state *S*, denote by S^+ the set of all edges *e* such that $S \cup e \in UP$. Consider, for example, the state *S*={3, 4} for the network on Fig. 4.a. This state is presented by super-state σ_{26} on Fig. 4.b. Clearly, S is a border state, and $S^+ = \{1, 2\}$.

Now we are ready to define the stationary transition rate more formally ([7]):

$$\Phi(N) = \sum_{S \in BD} P(S) \cdot \mu(S^+), \tag{15}$$

where *P(S)* is the probability of the border state *S* and $\mu(S^+) = \sum_{e \in S^+} \mu(e)$.

3.3. Estimating $\Phi(N)$

The number of border states in network is usually very large. Therefore we can only estimate the transition rate $\Phi(N)$. Let us explain here shortly the MC scheme for estimating $\Phi(N)$.

We consider the all-terminal criterion only. The case of *s-t* connectivity is more complicated *technically,* but not in principle. The difference between two criteria lies in the fact that for the all-terminal connectivity, all border states are located on the (*r*-1)-st level of the turnip, where *r* is the highest level. All states of this level contain exactly two connected components. In the case of the *s-t* connectivity, the border states may be located at each level of the turnip and may consist of any number of components. For example, we see on Fig. 4.b that for the *s-t* connectivity the state S={5} (which is associated with the super-state σ_{13}) is a border state and it consists of 3 components.

Consider now the sum for $\Phi(N)$ in (15), which is represented in the static form. We adopt the same parameterization which was defined in (2.1) and consider the Merging Process. So, we replace in the (15) the states by the super-states. Formally, after regrouping terms, we arrive at the following expression:

$$\Phi(N) = \sum_{\sigma \in BD} P(\sigma) \cdot \mu(\sigma^+). \qquad (16)$$

As it was explained earlier, the static probability $P(\sigma)$ may be interpreted as the probability that the Merging Process is in the super-state σ at moment t_0. Emphasize that we say "at moment t_0" but not "before the moment t_0". The reason for this is that in comparison with the UP-state, the border state (super-state) is not absorbing, i.e. our process may "jump" to the next (UP) state. Denote by $\xi(\sigma)$ the random time needed for the MP process to reach the super-state σ. We denoted earlier by $\xi(N)$ the random time for reaching the UP state. We have now for the $P(\sigma)$ the following formula:

$$P(\sigma) = P\{\xi(\sigma) \le t_0\} - P\{\xi(N) \le t_0\}. \qquad (17)$$

The latter expression means that the MP enters the border state σ before t_0 and remains there at least up to t_0. Introducing now the trajectories (in the same manner as it was in (2.3)), we get the following formula:

$$\Phi(N) = \sum_{u \in U} p(u)(P\{\xi(\sigma(u)) \le t_0 \mid u\} - P\{\xi(N) \le t_0 \mid u\}) \cdot \mu(\sigma(u)^+). \qquad (18)$$

We can do it because in the case of all-terminal connectivity, each trajectory *u* determines the unique border state $\sigma(u)$ (this does not take place for the *s-t* connectivity). So the sum in (18) has the form of (11) and therefore allows using the Monte Carlo scheme on turnip in the manner similar to (2.3): we simulate the trajectories with the probabilities *p(u)* defined by (9) and compute the probabilities in brackets using convolutions (10).

4. Conclusions

As mentioned, the above described approach was first suggested in [1] and developed later in the works [2-6]. It does a very successful use of the network combinatorics to develop a highly efficient Monte Carlo scheme. This approach has various aspects and problems: combinatorial algorithms on graphs, various computational problems (for example, calculating long convolutions), efficiency of the method in some cases, etc. All these aspects were examined in referreed works in exact and formal manner. Our first purpose was to explain the idea of the approach itself. We had no intention of reviewing formally complicated problems and technicalities.

Another purpose was to describe briefly the main advantages of the turnip, some of which were mentioned in this paper. Note that numerous simulation results given in [2, 3, 6] confirm the high performance of the turnip approach. The advantages are the following:

(a) The MP method (turnip) *eliminates in principle* the rare event phenomenon;

(b) The relative error of the method is bounded uniformly with respect to time;

(c) A single simulation *run* (a trajectory over the turnip) serves for as many values of t as needed, which means that in one Monte Carlo trial we can calculate the reliabilities of networks with same topologies and *different* vectors of edge down-probabilities;

(d) The turnip is very computationally efficient, especially for dense networks;

(e) For certain combinatorial cases of the networks, the turnip allows to carry out the algorithm complexity analysis;

(f) The "turnip" approach has many applications: estimation of T_{UP} and T_{DOWN}, Availability, differentiation formula for reliability (the gradient), etc.

(g) The approach is applicable to evaluating non-static (dynamic) reliability measures;

(h) The approach may be generalized from networks to binary monotonic systems.

We consider this approach to be very promising and further investigation in this field may yield many results in various directions.

References

1. Lomonosov, M.V., *Bernoulli scheme with closure*, Problems of Information Transmission (USSR), 10, 1974, 73-81.
2. T. Elperin, I. Gertsbakh, and M. Lomonosov, *Network reliability estimation using graph evolution models*, IEEE Transactions on Reliability, R-40, 1991, 572-581.
3. T. Elperin, I. Gertsbakh, and M. Lomonosov, *An Evolution Model for Monte Carlo Estimation of equilibrium Network Renewal parameters*, Probability in the Engineering and Informational Sciences, 6, 1992, 457-469.
4. M. Lomonosov, *On Monte Carlo estimates in network reliability*, Probab.Eng. Inf. Sci, 8, 1994, 245-264.
5. M. Lomonosov, Y. Shpungin, *Combinatorics of reliability Monte Carlo*, Algorithms&Complexity, 1999, 329-343.
6. Shpungin Y., *Combinatorial and Computational Aspects of Monte Carlo Estimation of Network Reliability*, PhD Thesis, Dept. of Mathematics and Computer Science, Ben-Gurion University of the Negev, 1997.
7. Gnedenko, B.V., ed., *Mathematical Aspects of Reliability Theory*, Moscow, "Radio I Sviaz" (in Russia), 1983.