PARAMETER ESTIMATION AND HYPOTHESES TESTING FOR NONHOMOGENEOUS POISSON PROCESS

I.B. Frenkel¹, I.B. Gertsbakh² and L.V. Khvatskin¹

¹Industrial Engineering and Management Department, Negev Academic College of Engineering, Beer-Sheva, 84100, Israel, E-mail: iliaf@nace.ac.il / khvat@nace.ac.il

²Department of Mathematics, Ben-Gurion University, P. O. Box 653, Beer-Sheva, 84105, Israel, E-mail: eliahu@cs.bgu.ac.il

Abstract

We consider a nonhomogeneous Poisson process (NHPP) with intensity function $\lambda(t)$. This function is parameterized in two forms: log-linear with $\lambda(t) = \exp(\alpha + \beta t)$ and Weibull-type $\lambda(t) = \alpha^\beta \beta t^{\beta-1}$. Parameter estimation is carried out by the well-known maximum likelihood procedure. We present its modus operandi and show that there is one and the only one solution to the maximum likelihood equation for the log-linear form of $\lambda(t)$.

For the case of the known intensity function, testing the hypothesis that the given sample path is a realization of NHPP can be carried out on the basis of the following well-known fact. Let $t_1, t_2, \ldots, t_n$ be the instants of event occurrences in NHPP. Consider the following time transformed process $(T_1, T_2, \ldots, T_n)$, where $T_i = \int_0^{t_i} \lambda(v)dv$. This process is a standard Poisson process (with $\lambda(t) = 1$) if the original process is NHPP. Therefore, the intervals between events in the transformed process form a sample of i.i.d. standard exponential random variables.

Let us consider the case of unknown intensity functions whose parameters are estimated from the sample path. It turns out that now the intervals between the adjacent transformed times $\hat{W}_i = \int_0^{\hat{\lambda}(v)} dv$ are not i.i.d. exponential random variables. We demonstrate that their sum, i.e. the value of $\hat{W}_n$, is always equal to $n$.

We propose the following computer-intensive procedure for testing the hypothesis that the given sample path belongs to NHPP with intensity function equal to the estimated $\hat{\lambda}(t)$:

1. Simulate $N$ trajectories of the NHPP with intensity function $\hat{\lambda}(t)$. Subject each trajectory to the above-described time transformation.

2. Compute for each simulation run the actual values of test statistics $S_1, S_2, S_3$ for the intervals between the adjacent transformed time instants. We denote $S_1$ as the Laplace-type statistic, $S_2$ as the Kolmogorov-Smirnov statistic and $S_3$ as the coefficient of variation of the transformed time intervals.
3. Based on 2, compute the upper and lower $\alpha$ - critical values for these statistics $s_i(\alpha), s_i(1-\alpha)$.

4. For the given sample path, compute the actual transformed time instants and the actual values of the above statistics $s_1, s_2, s_3$. Reject the hypothesis if at least one of these values fall outside the interval $[s_i(\alpha), s_i(1-\alpha)]$, for $i = 1,2,3$.

We present some numerical examples and show how the above described procedure works, for a sample path presented by M.J. Crowder et al (1991) and for a sample path generated by a renewal-type process with decreasing intervals between events.

**Key words:** Nonhomogeneous Poisson process, parameter estimation, time transformation, hypotheses testing.

1. Basic Facts About NHPP

**Definition 1.1.**

The counting process $\{N(t), t>0\}$ is said to be NHPP with intensity function $\lambda(t)$, $t>0$ ($\lambda(t)>0$) if

- $N(0) = 0$;
- $\{N(t), t\geq0\}$ has independent increments;
- $P(N(t+h) - N(t)\geq2) = o(h)$ as $h\to0$;
- $P(N(t+h) - N(t)\geq1) = \lambda(t)h + o(h)$ as $h\to0$.

**Definition 1.2.**

$$\Lambda(t) = \int_{0}^{t} \lambda(v) \, dv$$

(1)

is called the cumulative event rate or the mean value function of NHPP.

**Claim 1.1.**

$$P(N(t+s) - N(t) = k) = e^{-\Lambda(t,s)} \frac{[\Lambda(t,s)]^k}{k!}, \quad k \geq 0$$

(2)

where

$$\Lambda(t,s) = \Lambda(t,s) - \Lambda(t) = \int_{t}^{t+s} \lambda(v) \, dv.$$  

(3)

When $\lambda(t) = \lambda = const$, NHPP becomes a Poisson process with parameter $\lambda$, and $\Lambda(t) = \lambda t$.

Events happen at instants $t_1, t_2, \ldots, N(t)$ jumps by 1 at any event.

Note, that for the Poisson process, intervals between events $(0,t_1), (t_1,t_2), \ldots,(t_k,t_{k+1}),\ldots$ are i.i.d. exponential random variables with density $\lambda e^{-\lambda v}$.

The probabilistic characteristic of the intervals between adjacent events in the NHPP is given via the following claim.
Claim 1.2.

Suppose, that an event took place in NHPP at the instant $t$. Denote by $\tau_i$ the interval to the next event. Then

$$ P(\tau_i \leq x) = F_i(x) = 1 - e^{-\int^x_0 \lambda(v)dv} \quad (4) $$

and the corresponding density function is

$$ f_i(x) = \lambda(t+x) \exp \left[ \int^x_t \lambda(v)dv \right] \quad (5) $$

The random variable $\tau_i$ does not depend on the history of NHPP on $[0,t]$.

Note, that $\int^t_0 \lambda(v)dv = \Lambda(t)$ is the mean number of events on $[0,t]$ in NHPP.

2. The Likelihood Function for the NHPP

Suppose, we observe an NHPP on the interval $[0,T]$ and the events took place at the instants $0 < t_1 < t_2 < \ldots < t_n < T$.

Claim 2.1.

The likelihood function has the form

$$ \prod_{i=1}^n \int f(t_i) \exp \left[ -\int^T_0 \lambda(v)dv \right]. \quad (6) $$

Proof. Follows from the fact that

$$ Lik(t_1, \ldots, t_n; T) = f_0(t_1) \cdot f_{t_1}(t_2 - t_1) \cdot \ldots \cdot f_{t_{n-1}}(t_n - t_{n-1}) \cdot \left[ 1 - F_{t_n}(T - t_n) \right] $$

and from (4), (5).

If $T = t_n$ (this will be our standard assumption),

$$ Lik(t_1, \ldots, t_n; t_n) \equiv Lik(t_1, \ldots, t_n) = \prod_{i=1}^n \lambda(t_i) \cdot \exp \left[ -\int^t_0 \lambda(v)dv \right]. \quad (7) $$

3. Parameter Estimation for Special Cases of $\lambda(t)$

Two parameterizations are especially convenient for maximum likelihood estimations. The first form is the so-called log-linear with

$$ \lambda(t) = e^{\alpha + \beta t}, \quad (8) $$

and the second one is the Weibull form with
Both (8) and (9) were first introduced by Cox and Lewis (1966).

We will use the notation

\[ \lambda_t(t) = e^{\alpha + \beta t^{-1}} \]  
\[ \lambda_2(t) = \alpha_2 + \beta_2 t^{\beta_2} \]  

**Claim 3.1.**

The maximum likelihood estimators of \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are obtained as the solutions of the following systems of equations. (Note, that we assume \(t_n = T\)).

For \((\alpha_1, \beta_1)\) (log-linear form) it follows that

\[
\begin{cases}
    e^{\hat{\alpha}_1} = \frac{n\hat{\beta}_1}{e^{\hat{\beta}_1 t_n} - 1} \\
    \sum_{i=1}^{n} t_i + n\hat{\beta}_1^{-1} = \frac{nt_n}{1 - e^{-\hat{\beta}_1 t_n}}
\end{cases}
\]  

For \((\alpha_2, \beta_2)\) (Weibull form):

\[
\begin{cases}
    \hat{\alpha}_2 = \frac{n^{\frac{1}{\beta_2}}}{t} \\
    \hat{\beta}_2 = \log t_n - \frac{\sum_{i=1}^{n} \log t_i}{n}
\end{cases}
\]

Note that for the Weibull form there is an explicit solution for \((\hat{\alpha}_2, \hat{\beta}_2)\).

**4. Time Transformation in NHPP**

Let \(\{N(t), t \geq 0\}\) be NHPP with a known intensity function \(\lambda(t)\). Let us introduce the following transformation of time.

Events in NHPP occur at random instants \(T_1, T_2, \ldots, T_n, \ldots\). We define the transformation of time via the following relationship:

\[ W_i = \int_{0}^{T_i} \lambda(v)dv, \quad i = 1, 2, \ldots, n, \ldots \]  

In other words, in the transformed time events occur at the instants \(W_1, W_2, \ldots, W_n, \ldots\)

![Fig. 1. The time transformation](image-url)
The following fact arouses some special interest.

Claim 4.1.

Denote that \( \Delta_1 = W_1, \Delta_2 = W_2 - W_1, \ldots, \Delta_n = W_n - W_{n-1}, \ldots \)
\( \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots \) are i.i.d. random variables with distribution \( \text{Exp}(1) \). Hence, NHPP in the transformed time becomes the Poisson Process with intensity \( \lambda = 1 \).

For proof see, for example, Çinlar (1975), Theorem 7.4 and Meeker and Escobar (1998), Ch.16, Problem 16.14.

Claim 4.1. is very important for testing the hypothesis \( H_0 \) that the given process is NHPP with known intensity function \( \lambda(t) \): as suggested by Claim 4.1., consider the inter events intervals \( \Delta_1, \Delta_2, \ldots, \Delta_n, \ldots \) in the transformed time and check the hypothesis \( H'_0 \) that they are i.i.d. exponential random variables with parameter 1.

How to check the hypothesis that the given process is a NHPP when the intensity function is not known?

We suggest, first, to estimate the intensity function from the data and then to carry out the time transformation using the analogue of (14)
\[
\hat{W}_1 = \int_0^\tau \hat{\lambda}(v)dv,
\]
where \( \hat{\lambda}(v) \) is obtained by substituting maximum likelihood parameter estimates into the assumed form \( \hat{\lambda}_1(t) \) or \( \hat{\lambda}_2(t) \).

There is, however, a surprise here: \( \hat{W}_1 \) are not i.i.d. random variables anymore! Moreover, \( \hat{W}_1 + \hat{W}_2 + \ldots + \hat{W}_n \) always is equal to \( n \), i.e. this quantity is not random!

Claim 4.2.

For \( \hat{\lambda}_1(t) = e^{\hat{\alpha}_1 \hat{\beta}_1 t} \) and \( \hat{\lambda}_2(t) = \hat{\alpha}_2 \hat{\beta}_2 t^{\hat{\beta}_2-1} \),
\[
\hat{W}_1 + \hat{W}_2 + \ldots + \hat{W}_n = n.
\]

Proof.

Consider first the log-linear form. \( \hat{W}_1 + \hat{W}_2 + \ldots + \hat{W}_n \) is the transformed time for instant \( t_n \) in the original NHPP. Using (12) we obtain
\[
\hat{W}_1 + \ldots + \hat{W}_n = \int_0^t \hat{\lambda}_1(t) dt = \int e^{\hat{\alpha}_1 \hat{\beta}_1 t} dt = e^{\hat{\alpha}_1} \cdot \frac{1}{\hat{\beta}_1} \left[e^{\hat{\beta}_1 t} - 1\right] = e^{\hat{\alpha}_1} \cdot \frac{1}{\hat{\beta}_1} \cdot n = n.
\]

For the Weibull form from the first formula of (13) follows
\[
\hat{W}_1 + \ldots + \hat{W}_n = \int_0^t \hat{\lambda}_2(t) dt = \int_0^t \hat{\alpha}_2 \hat{\beta}_2 t^{\hat{\beta}_2-1} dt = \hat{\alpha}_2 \hat{\beta}_2 t_n^{\hat{\beta}_2} = n.
\]
Let $\lambda(t)$ is maximum likelihood estimator (MLE) of any other function. Will the previous property be also true for $\lambda(t)$?

5. Testing Hypothesis for NHPP

Suppose, we observed a counting process $\{N(t), t \geq 0\}$ on the interval $[0, t_n]$ and the times when events appear in this process are $(t_1, t_2, ..., t_n)$. Then we carry out the estimation of $\lambda(t)$ by assuming either the log-linear or the Weibull form of $\lambda(t)$. Denote the estimated intensity function by $\lambda(t)$. Define our assumption:

$\hat{H}_0$: The given process $\{N(t), t \geq 0\}$ is NHPP with intensity function $\lambda(t)$.

We suggest the following procedure for testing $\hat{H}_0$.

Simulate $M$ realizations of NHPP with intensity function $\lambda(t)$.

For each realization, let us carry out the time transformation (15). Denote by $\hat{W}^{(i)} = (\hat{W}_1^{(i)}, ..., \hat{W}_n^{(i)})$ the $i$-th realization of the transformed times, $i = 1, ..., M$.

Define several test statistics $S_1, S_2, S_3$ for the sample of the transformed times, for example:

$S_1$ – Kolmogorov-Smirnov distance between the ordered sample of in-between-events intervals

$$(\hat{\Delta}_1^{(i)}, \hat{\Delta}_2^{(i)}, ..., \hat{\Delta}_n^{(i)})$$

(16)

in transformed time and Exp(1);

$S_2$ – the Laplace-type statistic (also based on the intervals (16));

$S_3$ – the coefficient of variation (c.v.).

After carrying out the simulation stage, we will have the simulated values of all above statistics: $S_1^{(i)}, S_2^{(i)}, S_3^{(i)}, i = 1, 2, ..., M$. Determine the upper and lower critical values for these statistics.

Denote them as $S_1(\alpha), S_1(1-\alpha), S_2(\alpha), S_2(1-\alpha), S_3(\alpha), S_3(1-\alpha)$ for $\alpha = 0.005, \alpha = 0.025, \alpha = 0.05$.

For our given single realization of the hypothetical NHPP with events at the instants $(t_1, t_2, ..., t_n)$, we investigate the following operations:

- Carry out the time transformation $\hat{V}_i = \int_0^{t_i} \hat{\lambda}(t)dt$, $i = 1, ..., n$ and compute the intervals between adjacent events in the transformed time:

$$\hat{\delta}_1 = \hat{V}_1, \hat{\delta}_2 = \hat{V}_2 - \hat{V}_1, ..., \hat{\delta}_n = \hat{V}_n - \hat{V}_{n-1} .$$

(18)

- For the sample $(\hat{\delta}_1, ..., \hat{\delta}_n)$, compute the values $S_1^*, S_2^*, S_3^*$ of the above defined statistics.
- Compare $S_1^*, S_2^*, S_3^*$ with the upper and lower critical values $S_1(\alpha), S_1(1-\alpha), S_2(\alpha), S_2(1-\alpha), S_3(\alpha), S_3(1-\alpha)$.

Reject $\tilde{H}_0$ if one of the statistics $S_1^*, S_2^*, S_3^*$ doesn’t suit the intervals $[S_1(\alpha), S_1(1-\alpha)]$ or $[S_2(\alpha), S_2(1-\alpha)]$, or $[S_3(\alpha), S_3(1-\alpha)]$.

**Remark.**

Consider a NHPP with $\tilde{\lambda}(t)$ estimated from data. Denote it as $\hat{\lambda}(t)$. Then the estimate of $\tilde{\Lambda}(t_n) = \int_{0}^{t_n} \hat{\lambda}(x) dx$ is equal to $n$, as we have seen for two our $\tilde{\lambda}(t)$ functions. But $\hat{\lambda}(t_n)$ is the MLE of the mean number of events on $[0, t_n]$. Therefore for our two $\tilde{\lambda}(t)$ functions, $\hat{\lambda}(t_n)$ is an unbiased estimator of the mean number of events because $\hat{\lambda}(t_n) = n$, and we have exactly $n$ events on $[0, t_n]$.

### 6. Testing the NHPP Hypothesis on Experimental and Simulated Data

#### 6.1. Experimental Data

Crowder et al. (1991), on the page 161 gives the data on failures of a diesel engine of USS Grampus. This data was fitted to $\tilde{\delta}(t) = 1.06$, with the total number of observed events $n = 56$.

The simulated critical values of the Laplace-type statistic $S_1$ were: $S_1(0.01) = -0.98, S_1(0.99) = 1.19$. The observed $\tilde{S}_1 = 0.86$.

Similarly, the critical values of Kolmogorov-Smirnov (K-S) statistic are $S_2(0.01) = 0.03, S_2(0.99) = 0.15$. The observed K-S statistic is $\tilde{S}_2 = 0.07$.

The critical values of the coefficient of variation (c.v.) statistic are $S_3(0.01) = 0.85, S_3(0.99) = 1.17$. The observed $\tilde{S}_3 = 1.03$.

Thus, all three criteria do not contradict the NHPP hypothesis. Also, the authors claim, that this data set comes from NHPP.

At the page 173 Crowder et al. (1991) presents another data set on failure times for the engine of USS Halfbeak with $n = 71$ observed system failures. The authors express doubts that this data set comes from NHPP. Our tests reveal the following: Two of three statistics ($\tilde{S}_1$ and $\tilde{S}_3$) fall outside the corresponding $[0.01, 0.99]$ simulated intervals for these statistics. Therefore, by our method we would claim that the data are not from NHPP.

#### 6.2 Simulated Data

We checked our testing procedure on a series of realizations of random processes which were not NHPP. In the typical simulated random process the events take place at the instants

$$X_1 = t_1, X_1 + X_2 = t_2, ..., X_1 + ... + X_n = t_n$$
where $X_1, X_2, ..., X_n$ are independent random variables, having the following structure:

$$X_i = Y_i + Z_i.$$ 

The random variables $Y_i$ and $Z_i$ are independent, and $Y_i \sim \text{Exp}(\lambda_i)$ and $Z_i \sim \text{U}(a_i, b_i)$. In other words, the intervals between events in the original process were sums of exponential and uniform random variables.

For all simulation runs, $n = 100$. For the sake of convenience we denote the simulated random processes as $\xi_k(t), k = 1, 2, ..., 7$. Below is the description of these processes.

$\xi_1(t)$: all $Z_i$ are zero, $Y_i \sim \text{Exp}(\lambda_i)$ with $\lambda_i = \frac{1}{\sqrt{100i} - \sqrt{100(i-1)}}, i = 1, 2, ..., 100$.

This is a sequence of independent exponential random variables with decreasing mean values.

$\xi_2(t)$: $Y_i \sim \text{Exp}(0.9\lambda_i)$ and $Z_i \sim \text{U}(-0.1+0.1\lambda_i, 0.1+0.1\lambda_i)$.

$\xi_3(t)$: $Y_i \sim \text{Exp}(0.8\lambda_i)$ and $Z_i \sim \text{U}(-0.1+0.2\lambda_i, 0.1+0.2\lambda_i)$.

$\xi_4(t)$: $Y_i \sim \text{Exp}(0.7\lambda_i)$ and $Z_i \sim \text{U}(-0.1+0.3\lambda_i, 0.1+0.3\lambda_i)$.

$\xi_5(t)$: $Y_i \sim \text{Exp}(0.9\lambda_i)$ and $Z_i \sim \text{U}(-0.2+0.1\lambda_i, 0.2+0.1\lambda_i)$.

$\xi_6(t)$: $Y_i \sim \text{Exp}(0.9\lambda_i)$ and $Z_i \sim \text{U}(-0.3+0.1\lambda_i, 0.3+0.1\lambda_i)$.

$\xi_7(t)$: all $Y_i$ are zero, $Z_i$ are nonrandom quantities equal to

$$V_i = \sqrt{100i} - \sqrt{100(i-1)}, \ i = 1, 2, ..., 100;$$

Thus, the first process is a sequence of exponential distributed variables, and the last one is a sequence of deterministic random variables. In-between, we have exponential random variables perturbed with uniform random variables.

The obtained test results are summarized in Table 1. Analyzing the last column of this table denoted as “Conclusion”, we can sum up the results.

The sequence of purely exponential random variables is not recognized by our tests as a not NHPP (formally, this sequence is not NHPP). Slightly perturbed sequence of exponential random variables ($\xi_2(t)$ and $\xi_5(t)$) are also not recognized by our tests as being not NHPP. Other processes are recognized by our tests as not being NHPP, at least by one of the three test statistics. The c.v. statistic performs approximately as good as the K-S based statistic. The last process $\xi_7(t)$ was recognized by two statistics as being not NHPP. Since, it is designed to recognize the deviation of uniform distribution. In our case, the intervals between events in the transformed time divide the interval $[0, n]$ into equal parts, and the Laplace-type statistic does not recognize this as a deviation from uniformity.
Table 1. The obtained Test Results

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<th>Statistics</th>
<th>Conclusion</th>
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