RESAMPLING ESTIMATORS OF HIERARCHICAL RELIABILITY SYSTEMS

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Abstract

This paper illustrates application of Hierarchical Resampling method for simulation of hierarchical sequential-parallel reliability systems in the case when the same sample population for some elements is used. Modifications of Hierarchical Resampling method are proposed for given task. Formulas for variance of estimators are given. Obtained estimators are compared with traditional ones calculated using empirical distribution functions. It is shown that considered approach can be a good alternative to traditional one.

1. Introduction

Resampling method is intensive statistical method which can be used very efficiently in the case of small initial samples. In this case, we cannot obtain good estimators using traditional methods. Resampling can be used as alternative approach in this case.

"Resampling" means that values of initial variables of simulated system are not generated using random number generator, but extracted, with replacement, directly from primary samples. This approach can be effective because it uses initial data in different combinations.

2. Structure of considered system

In order to define the considered system let us define a term "structured system". Structured system contains units of two types:

- single element,
- structured system (also called subsystem).

Let us define single element as element which can be in two states:

- working,
- failure.

Structured system also can be in working or failure statement. In order to define the state of structured system, we must know its type.

There are two types of structured systems:

- sequential structured system;
- parallel structured system.

In the case of sequential structured system all units of the system are connected sequentially. Sequential system works if all units of the system works (including all single elements and subsystems).

In the case of parallel system all units of the system are connected in parallel. Parallel system works if at least one unit of system works (including all single elements and subsystems).

If we start from some structured system (let us define it as top-level structured system) and examine its units, we can find subsystems among of them. These are subsystems of level 2. Examining each of them, we will obtain subsystems of level 3, etc. It is clear that units of last level systems (leaves of hierarchy) all are single elements.

Let us change order of units in each structured system to make the following:
first are coming single elements, then subsystems.

Now let us define the system more formally. Let the system contains r levels of hierarchy. Let us consider the level l. Let there are m_l subsystems on level l. Let S_{il} be i-th by order system on l-th level, i=1...m_l, l=1...r.

Let the system S_{il} contains $k1_{il}$ single elements and $k2_{il}$ subsystems. Let random variable $\mathbf{X}_{ilj}$ define working time of j-th element in the system S_{il}, j=1...$k1_{il}$.

Let single elements within one system are identical be their properties: distribution of $\mathbf{X}_{ilj}$ is identical for all j, j=1...$k1_{il}$.

We will define $f_{il}(x)$ corresponding probability density function and $F_{il}(x)$ the distribution function. So $F_{il}(x)=P\{\mathbf{X}_{ilj} \le x\}$ for all j.

Let $\mathbf{Y}_{il}$ be a working time of system S_{il}. We will define $g_{il}(x)$ corresponding probability density function and $G_{il}(x)$ the distribution function. So $G_{il}(x)=P\{Y_{il} \le x\}$. Let us also define the reliability function $R_{il}(x)=1-F_{il}(x)$.

All distributions $f_{il}(x)$ are unknown, only sample population $H_{il} = (x_{il1}, x_{il2}, ..., x_{i\ln_{il}})$ of size n_{il} is available for each system S_{il} as its single elements working times. Properties of one system elements are identical, so only one sample for all elements inside the system is available.

Our goal is for given time t estimate $R(t)=R_{11}(t)$. It is reliability function of all the system, or, in other words, the probability that system will work more than time t.

3. Traditional approach

In traditional approach, we use empirical distribution functions to estimate R(t).

First, let us consider a system S_{il} that doesn't have subsystems, $k2_{il}=0$. Let $\#(x_{ilj} \le t)$ be number of elements from sample H_{il} that are less or equal of t. Then

$$F_{il}^*(t) = \frac{1}{n_{il}} \#(x_{ilj} \le t). \tag{1}$$

It is easy to see that for sequential system estimator of $G_{il}(t)$ is

$$G_{il}^*(t) = 1 - (1 - F_{il}^*(t))^{k1_{il}}.$$

For parallel system estimator is

$$G_{il}^*(t) = F_{il}^*(t)^{k1_{il}}.$$

It can be shown (Fioshin, 2000) that estimators are biased and in both cases $E(G^*_{il}(t)) \neq G_{il}(t)$. It is clear because we multiply by itself the same empirical distribution function but work of elements is independent.

Of course, if estimator of simple system is biased, also biased will be estimator of hierarchical system.

4. Resampling simulation

Now we propose an alternative approach that we call Resampling approach. Instead of estimating $F_{il}(t)$ we extract values from initial samples and use them as single elements working times. This allows to avoid bias if in one realization values from the sample are extracted without replacement.

In order to realize the proposed approach we must suppose that size of samples H_{il} are not less that number of simple elements $k1_{il}$:

$$| H_{il} | \leq k1_{il}.$$

We start our simulation from systems S_{il} that don't have subsystems ($k2_{il}=0$). In this case we extract without replacement $k1_{il}$ elements from H_{il}. They represent working times of simple elements. Then we can act in two ways:

- if the system is sequential we take minimum of these times,
- if the system is parallel then we take maximum of these times.

So we obtain whole system working time.

Repeating this procedure m_{il} times we will form a sample H'_{il} of given system working times.

Now let us consider a case when a system has subsystems. Let us denote Sub_{il} a set of all subsystems on a level $l+1$ of a system S_{il}. It is clear that $|Sub_{il}|=k2_{il}$. Let samples H'_{jl+1} of all subsystems $S_{jl+1} \in Sub_{il}$ working times are formed.

First we extract, without replacement, $k1_{il}$ elements from sample H_{il}. They are realizations of simple elements working times. Then we extract one element from each of $k2_{il}$ samples $H'_{jl+1} \in Sub_{il}$ of subsystems working times. Then we act as following:

- if the system is sequential we take minimum of all these times,
- If the system is parallel then we take maximum of all these times.

So we get one realization of system S_{il} working time. Repeating this procedure m_{il} times we form a sample H'_{il} of the system S_{il} working times.

Repeating this procedure recurrently we obtain sample $H'_{11}=(y_{111}, y_{112}, ..., y_{11m})$ of size $m=m_{11}$ of top-level system working times. Now as an estimator of R(t) we propose simply to take

$$R^*(t) = \frac{1}{m} \#(y_{11j} \geq t), j = 1...m. \quad (2)$$

This estimator is unbiased. Now we are interested in its variance

$$D = D\ R^*(t). \quad (3)$$

5. Variance calculation

Variance of this estimator also will be calculated recurrently.

First let us start from a system S_{il} which doesn't have a subsystems. Let us examine two elements of sample H'_{il} of the system working times. In order to calculate variance we will replace elements of this sample by 0-s and 1-s. 0 will mean that corresponding element y_{ilj} is less than t, but 1 means that this element is not less than t. Let us form also a sample $H''_{il}=(z_{il1}, z_{il2}, \ldots , z_{ilm}\}$, where $z_{ilj}=1$ if $y_{ilj} \geq t$ and $z_{ilj}=0$ elsewhere.

Let us denote Cov_{il} covariance of two elements of sample H''_{il}:

$$Cov_{il} = Cov(z_{ilj}, z_{ilk}).$$

Clear that

$$\begin{aligned} Cov_{il} &= E(z_{ilj} \cdot z_{ilk}) - E(z_{ilj})E(z_{ilk}) = \\ &= P\{y_{ilj} \geq t \,\&\, y_{ilk} \geq t\} - P\{y_{ilj} \geq t\}^2. \end{aligned}$$

Probability $P\{y_{ilj} \geq t\}$ can be calculated for two types of systems as following:

- The case of sequential system:

$$P\{y_{ilj} \geq t\} = (1 - F_{il}(t))^{k1_{il}}.$$

- The case of parallel system:

$$P\{y_{ilj} \geq t\} = 1 - F_{il}(t)^{k1_{il}}.$$

Now let us go to probability $P\{y_{ilj} \geq t \,\&\, y_{ilk} \geq t\}$ calculation.

For its calculation let us take hypothesis $A\{\alpha\}$: "for calculation of y_{ilj} and y_{ilk} directly α common elements from sample H_{il} were taken", $\alpha = 1, \ldots ,n_{il}$.

Let us denote $P\{ y_{ilj} \geq t \,\&\, y_{ilk} \geq t \mid A\{\alpha\}\}$ the corresponding conditional probability under this hypothesis. Now we can calculate this probability for both types of systems:

Sequential system. This probability can be calculated as following:

$$P\{y_{ilj} \geq t \,\&\, y_{ilk} \geq t \mid A(\alpha)\} = (1 - F_{il}(t))^{2k1_{il}-\alpha}. \tag{4}$$

- Parallel system. Here we must consider two non-intersected events:
  1) among common α elements there is at least one that $\geq$ t;
  2) among common α elements all elements are $\leq$ t, but in both samples there are at least one element that is $\geq$ t.

$$\begin{aligned} P\{y_{ilj} \geq t \,\&\, y_{ilk} \geq t \mid A(\alpha)\} &= 1 - F_{il}(t)^{\alpha} + F_{il}(t)^{\alpha}(1 - F_{il}(t))^{2k1_{il}-\alpha} = \\ &= 1 - F_{il}(t)^{2k1_{il}-\alpha}. \end{aligned} \tag{5}$$

The probability of event $A\{\alpha\}$ is given in both cases by hyperheometrical distribution:

$$P\{A\{\alpha\}\} = \frac{C_{k1_{il}}^{\alpha} C_{n_{il}-k1_{il}}^{k1_{il}-\alpha}}{C_{n_{il}}^{k1_{il}}}. \tag{6}$$

This allows to calculate recurrently all covariances C_{il}. Last covariance which will be calculated will be $C_{11} = Cov(z_{11j}, z_{11k})$.

Replacing in (2) y_{11j} by z_{11j} we get

$$R^*(t) = \frac{1}{m}\sum_{j=1}^{m} z_{11j}. \tag{7}$$

Then its variance can be calculated as

$$\begin{aligned} D\ R^*(t) &= \frac{1}{m^2} m\ D(z_{11j}) + 2\frac{m(m+1)}{2} C_{11} = \\ &= \frac{D(z_{11j})}{m} + \frac{m-1}{m} C_{11}. \end{aligned} \tag{8}$$

Variance $D(z_{11j})$ can be simply calculated because z_{11j} represents result of Bernoulli trials with success probability R(t):

$$D(z_{11j}) = R(t)(1 - R(t)). \tag{9}$$

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