PROPERTIES OF GUARANTEED REACHABLE SETS FOR LINEAR DYNAMIC SYSTEMS UNDER UNCERTAINITIES WITH INTERMEDIATE CORRECTION POINTS

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In the paper we consider the problem of investigation and constructive description of the reachable sets for linear dynamic systems under unknown but bounded uncertainties over feasible controls that are allowed to be corrected in a given set of correction moments. It is showed that construction of the reachable sets under considered class of feasible uncertainties can be reduced to solving a multilevel min-max optimisation problem with respect to finite dimensional decision variables.

It is proved that adding one new correction moment leads to an extension of the set of system states that can be guaranteed reached at a terminal moment from a given initial position over a feasible control strategy. Numerical example illustrates theoretical results. Some rules for construction external and internal approximations for the reachable sets are discussed.

Keywords: reachability set, control system, bounded uncertainties

1. Introduction

One of the central problems in mathematical optimal control theory is the problem of constructive description and determination of the reachability set for a given controlled system, i.e. the set of all system states that can be reached at a final moment from a given initial state with some feasible control function. There exists a wide range of literature devoted to investigation and construction of the reachability sets [1–5]. Such investigations are important both from theoretical and practical points of view since they allow us to evaluate potentialities of different dynamic system. For systems with uncertainties, the reachability sets give an opportunity to evaluate system states spread under these uncertainties.

The problem of computation of the reachability set can be interpreted as a special problem of theory of differential games. In such problem statement, both control and uncertainty are considered as actions of the first and the second players respectively but for all that the second player has no certain objects. However, if one follows a guaranteed approach then the aim of the second is to do damage to the first player [2, 4].

In many practical problems, the exact determination of the reachability set is a very difficult task. Therefore, in many applications, the exact reachability set is replaced by internal or external approximation which has a more simple structure, for example by some polyhedron or ellipsoid. At present there is a number of methods for constructing approximation of reachable sets of dynamic controlled systems. In particular, much attention is paid to ellipsoidal approximations [2–5].

In this paper, we investigate properties of the reachable sets for dynamic controlled systems with uncertainties. It is supposed that control function can be corrected in some given moments on the base of available information about current system state. It is well-known [6–8] that these allowed control corrections lead to extended reachable set compared to approaches that do not use control corrections. The price that must be paid for this benefit is that the computational demands for solving corresponding multilevel min-max optimisation problems may be very high. In the paper we show that arising min-max optimisation problem with respect to functional decision variables may be reduced to a multilevel min-max with respect to finite dimensional decision variables. This allows us to give constructive description of the reachable set and investigate its properties. Theoretical results are illustrated by a numerical example.

2. Problem Statement. Definitions

Consider a dynamic system that behaviour is described by the following system of ordinary differential equations

$$\dot{z}(t) = Az(t) + bu(t) + gw(t), \ t \in T = [0, t_*], \ z(0) = z_0, \ (1)$$

where $z(t) \in \mathbb{R}^n$ is a system state, $u(\cdot) = (u(t), t \in T) \in U$, is a scalar control function, $w(\cdot) = (w(t), t \in T) \in W$ is unknown disturbance, matrix $A \in \mathbb{R}^{n \times n}$, and vectors $b, g \in \mathbb{R}^n$, and moments
$t_0, t_*$ are considered to be given. The set of feasible controls $U$ and the set of feasible disturbances $W$ are defined as follows:

$U = U(T) = \{ u(\cdot) = (u_s(\cdot), s = 1, \ldots, N + 1) : u_s(\cdot) \in U_s, s = 1, \ldots, N + 1 \}$,

$W = W(T) = \{ w(\cdot) = (w_s(\cdot), s = 1, \ldots, N + 1) : u_s(\cdot) \in W_s, s = 1, \ldots, N + 1 \}$,

$U_s = \{ u_s(\cdot) = (u(t), t \in T_s) \in L_2(T_s) : \int_{T_s} u^2(t) dt \leq r_s \}$,

$W_s = \{ u_s(\cdot) = (w(t), t \in T_s) \in L_2(T_s) : \int_{T_s} w^2(t) dt \leq v_s \}$,

where $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = t_*$. (2)

Denote by $z(t|x, u(\cdot), w(\cdot))$, $t \geq \tau$, the trajectory of the system

$\dot{z}(t) = Az(t) + bu(t) + gw(t)$, $t \geq \tau$, $z(\tau) = x$

generated by an initial state $z(\tau) = x$, by a control $u(\cdot)$ and by a disturbance $w(\cdot)$.

Pair $(z_s, \tau_s)$ of $n$-vector $z_s$ and a moment $\tau_s \in T$ is called a position of system (1). We say that system is in a position $(z_s, \tau_s)$ if at the moment $\tau_s$ system state is $z_s$.

Let us consider some subset of time instances from a given set (2)

$\tau_i = \tau_m$, $i = 0, \ldots, i_s$, $m_0 = 0 < m_1 < m_2 < \ldots < m_{i_s} < N + 1 = m_{i_s+1}$. (3)

For $i = 0, 1, \ldots, i_s$, denote

$I_i = [m_i + 1, m_i + 2, \ldots, m_{i_s+1}]$,

$U^{(i)} = \{ u^{(i)}(\cdot) = (u(t), t \in [\tau_i, \tau_{i+1}]) : u_s(\cdot) \in U_s, s \in I_i \}$,

$W^{(i)} = \{ w^{(i)}(\cdot) = (w(t), t \in [\tau_i, \tau_{i+1}]) : w_s(\cdot) \in W_s, s \in I_i \}$.

Given $\mu > 0$, $Q = Q^T > 0$, $Q \in \mathbb{R}^{n \times n}$, denote $B(x|\mu, Q) = \{ z \in \mathbb{R}^n : \| x - z \|^2 \leq \mu \}$.

Definition 1. A system state $x \in \mathbb{R}^n$ is said to be $(\mu_s, Q_s)$ - guaranteed reachable (at the terminal moment $t = t_*$) from a position $(z_{i_s}, \tau_{i_s})$, $z_{i_s} \in \mathbb{R}^n$, $0 \leq \tau_{i_s} < t_*$, if there exists a control $u^{(i_s)}(\cdot) \in U^{(i_s)}$, such that $Z_{i_s}(z_{i_s}, u^{(i_s)}(\cdot)) \subset B(x|\mu_s, Q_s)$.

Here and in what follows $Z_i(z_{i_s}, u^{(i)}(\cdot)) = \{ z = z(\tau_{i_s} | z_{i_s}, \tau_{i_s}, u^{(i)}(\cdot), w^{(i)}(\cdot)) : w^{(i)}(\cdot) \in W^{(i)} \}$.

Definition 2. A system state $x \in \mathbb{R}^n$ is said to be $(\mu_s, Q_s)$ - guaranteed reachable (at the terminal moment $t = t_*$) form a position $(z_{i_{s-1}}, \tau_{i_{s-1}})$, $z_{i_{s-1}} \in \mathbb{R}^n$, $0 \leq \tau_{i_{s-1}} < \tau_{i_s}$, subject to one correction moment $\tau_{i_s}$, if there exists a control $u^{(i_{s-1})}(\cdot) \in U^{(i_{s-1})}$, such that the state $x$ is $(\mu_s, Q_s)$ - guaranteed reachable from all positions $(z, \tau_{i_s})$, $z \in Z_{i_{s-1}}(z_{i_{s-1}}, u^{(i_{s-1})}(\cdot))$.

Definition 3. For $0 \leq i < i_s$, a system state $x \in \mathbb{R}^n$ is said to be $(\mu_s, Q_s)$ - guaranteed reachable form a position $(z_{i}, \tau_{i})$, $z_{i} \in \mathbb{R}^n$, $0 \leq \tau_{i}$, subject to $(i_s - i)$ correction moments $\tau_{i+1}, \tau_{i+2}, \ldots, \tau_{i_s}$, if there exists a control $u^{(i)}(\cdot) \in U^{(i)}$, such that the state $x$ is $(\mu_s, Q_s)$ - guaranteed reachable subject to $(i_s - i - 1)$ correction moments $\tau_{i+2}, \ldots, \tau_{i_s}$ from all positions $(z, \tau_{i})$, $z \in Z_i(z_{i_s}, u^{(i)}(\cdot))$. 


Denote by \( X_s(z_i | \tau_i) \) the set of all system states that are \((\mu_s, Q_s)\) – guaranteed reachable (without correction) from position \((z_i, \tau_i)\). For \( i = i_s - 1, \ldots, 0 \), denote by \( X_s(z_j | \tau_j, \tau_{j+1}, \ldots, \tau_i) \) the set of all system states that are \((\mu_s, Q_s)\) – guaranteed reachable from position \((z_j, \tau_j)\) subject to the intermediate correction moments \( \tau_{j+1}, \ldots, \tau_i \).

It is evident that \( x \in X_s(z_i | \tau_i, \ldots, \tau_i) \) if and only if there exists a control \( u^{(i)}(\cdot) \in U^{(i)} \) such that \( x \in X_s(z_i | \tau_i, \ldots, \tau_i) \) for all \( z \in Z_s(z_i, u^{(i)}(\cdot)) \), i.e.
\[
X_s(z_j | \tau_j, \ldots, \tau_i) = \{ x \in \mathbb{R}^n : \exists u^{(i)}(\cdot) \in U^{(i)} \text{ such that } \forall z \in Z_s(z_j, u^{(i)}(\cdot)) \}.
\]

The aims of the paper are the following

A. to give constructive description of the set
\[
X_s(z_0 | \tau_0, \ldots, \tau_i),
\]
i.e. to describe the set of all system state that are \((\mu_s, Q_s)\) – guaranteed reachable at the terminal moment \( t_* \) from a given initial position \((z_0, \tau_0) = 0\) with intermediate correction moments \( \tau_1, \ldots, \tau_i \).

B. to investigate the dependence of the set (4) on a set of correction time instances (3).

3. Different Presentations of the Reachable Set with a Fixed Set of Correction Moments

In this section, we suppose that the set of correction moments (3) is fixed. Let us describe different presentations of the reachable sets \( X_s(z_j | \tau_j, \ldots, \tau_i) \), \( i = 0, \ldots, i_s \).

One can easily check that the following proposition holds true.

**Proposition 1.** The sets \( X_s(z_j | \tau_j, \ldots, \tau_i) \), \( i = 0, \ldots, i_s \), admit the following presentations
\[
\bar{\mathcal{F}}_s(x | z_i) = \min_{u^{(i)}(\cdot) \in U^{(i)}} \max_{w^{(i)}(\cdot) \in W^{(i)}} \mathcal{F}_{i+1}(x | \bar{\mathcal{G}}_{i+1}(z_i, u^{(i)}(\cdot), w^{(i)}(\cdot))), \quad i = 0, 1, \ldots, i_s - 1,
\]
with the initial condition
\[
\bar{\mathcal{F}}_s(x | z_i) = \min_{u^{(i)}(\cdot) \in U^{(i)}} \max_{w^{(i)}(\cdot) \in W^{(i)}} \mathcal{F}_{i+1}(x | \bar{\mathcal{G}}_{i+1}(z_i, u^{(i)}(\cdot), w^{(i)}(\cdot))) - x_0^2.
\]
Here \( \bar{\mathcal{G}}_{i+1}(z_i, u^{(i)}(\cdot), w^{(i)}(\cdot)) = F(t_{i+1}, \tau_i)z_i + \sum_{s \in \mathbb{I}_{i+1}} \int F(t_{i+1}, t)(bu_s(t) + gw_s(t))dt, \quad i = 0, 1, \ldots, i_s \), \( F(t, \tau) \) is the fundament solution matrix for the system \( \dot{x} = Ax \).

**Remark 1.** The relations (5), (6) imply that
\[
\bar{\mathcal{F}}_s(x | z_i) = \min_{u^{(i)}(\cdot) \in U^{(i)}} \max_{w^{(i)}(\cdot) \in W^{(i)}} ||F(t, \tau)z_i + \sum_{s \in \mathbb{I}_{i+1}} \int F(t, t)(bu_s(t) + gw_s(t))dt, \quad s = 1, \ldots, i_s \).
\]

Note that, in this min-max optimisation problem, decision variables are functions \( u^{(i)}(\cdot), w^{(i)}(\cdot), s = 1, \ldots, i_s \).

Here and in what follows the set of indices \( s = q, q + 1, \ldots, m \) is considered to be empty if \( m < q \).

Denote
\[
G_s = \int_{t_s}^{t_*} (F(t, t)b)(F(t, t)b)^T dt \in \mathbb{R}^{n \times n}, \quad Q_s = \int_{t_s}^{t_*} (F(t, t)g)(F(t, t)g)^T dt \in \mathbb{R}^{n \times n},
\]
\[
\Psi_s = \{ \psi \in \mathbb{R}^n : \psi^T G_s \psi \leq r_s \}, \quad \Phi_s = \{ \phi \in \mathbb{R}^n : \phi^T Q_s \phi \leq v_s \}, s = 1, \ldots, N + 1.
\]
Theorem 1. Functions (5), (6) can be presented in the form
\[ F(x | z_i) = \gamma_i(x | F(t_i, \tau_i)z_i) \quad i = 0, 1, \ldots, i_* \] (7)
where functions \( \gamma_i(x | d), i = 0, \ldots, i_* \), are determined by the following recursive relations
\[ \gamma_i(x | d) = \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \max \gamma_{i+1}(x | d + \sum_{s \in l_i} (G, y_s + Q, \phi_s)), \quad i = 0, 1, \ldots, i_* - 1, \] (8)
with the initial condition
\[ \gamma_i(x | d) = \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \max \| d + \sum_{s \in l_i} (G, y_s + Q, \phi_s) - x \|_{Q_i}^2. \] (9)

Remark 2. The relations (8), (9) imply that
\[ \gamma_i(x | F(t_i, \tau_i)z_i) = \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \max \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \ldots \min_{\psi_i, i \in I_i, \phi_i \in \Phi_i} \| F(t_i, \tau_i)z_i + \sum_{s = m_i + 1}^{N+1} (G, y_s + Q, \phi_s) - x \|_{Q_i}^2 \]
subject to \( \psi_i^T G, y_s \leq r_s, \phi_i^T Q, \phi_s \leq v_s, s = m_i + 1, \ldots, N + 1. \)

Remark 3. Note that the functions \( \gamma_i(x | d), i = 0, \ldots, i_* \), are simpler than the functions \( \gamma_i(x | z_i), i = 0, \ldots, i_* \), since for construction of the functions \( \gamma_i(x | d) \), one needs to solve min-max (finite dimensional) optimisation problems with respect to the set of \( n \)-vectors \( \psi_i, \phi_i, s \in I_i, k = i, \ldots, i_* \), while for constructing the functions \( \gamma_i(x | z_i) \), one has to solve min-max (infinite dimensional) optimisation problems with respect to the set of functions \( u_s(\cdot), w_s(\cdot), s \in I_i, k = i, \ldots, i_* \).

Proof of Theorem 1. For \( i = 0, 1, \ldots, i_* \), let us consider the sets
\[ Z_s(i) = \{ z \in R^n : z = \sum_{s \in l_i} \int_{T_i} F(t_i, t)gw(t) dt, w_s(\cdot) \in W_s, s \in I_i \}, \] (10)
\[ \tilde{Z}_s(i) = \{ z \in R^n : z = \sum_{s \in l_i} Q, \phi_s, \phi_s^T Q, \phi_s \leq v_s, s \in I_i \}, \] (11)
and show that
\[ Z_s(i) = \tilde{Z}_s(i), i = 0, 1, \ldots, i_* \] (12)

It is evident that the sets \( Z_s(i) \) and \( \tilde{Z}_s(i) \) can be written in the equivalent form:
\[ Z_s(i) = \{ z \in R^n : z = \sum_{s \in l_i} \Delta z_s, \Delta z_s \in \Delta Z_s, s \in I_i \}, \]
\[ \tilde{Z}_s(i) = \{ z \in R^n : z = \sum_{s \in l_i} \Delta z_s, \Delta z_s \in \Delta \tilde{Z}_s, s \in I_i \}, \]
where \( \Delta Z_s = \{ \Delta z \in R^n : \Delta z = \int_{T_i} F(t_i, t)gw(t) dt, w_s(\cdot) \in W_s \}, s = 1, \ldots, N + 1, \)
\( \Delta \tilde{Z}_s = \{ \Delta z \in R^n : \Delta z = Q, \phi_s, \phi_s^T Q, \phi_s \leq v_s \}, s = 1, \ldots, N + 1 \).

It follows from Lemma 1 [9, page 134] that the equalities \( \Delta Z_s = \Delta \tilde{Z}_s, s = 1, \ldots, N + 1 \), take place. Evidently, these equalities imply (12).

Similarly one can show that the equalities take place
\[ Z_{ss}(i) = \tilde{Z}_{ss}(i), i = 0, 1, \ldots, i_* \] (13)
with
\[ Z_{ss}(i) = \{ z \in R^n : z = \sum_{s \in l_i} \int_{T_i} F(t_i, t)bu_s(i) dt, u_s(\cdot) \in U_s, s \in I_i \}, \] (14)
\[ \hat{Z}_n(i) = \{ z \in \mathbb{R}^n : z = \sum_{s \in I_i} G_s \psi_s \psi_s^T G_s \psi_s^T \leq r_s, s \in I_i \} \]  

(15)

Let us prove the equality (7) for \( i = i_* \), i.e., let us show that

\[ \min_{u^{(i*)} \in \mathbb{R}^{(i)}} \max_{w^{(i*)} \in \mathbb{R}^{(i)}} \| F(t_s, \tau_{i_*}) z_{i_*} + \sum_{s \in I} F(t_s, t)(bu_s(t) + gw_s(t)) dt - x \|^2_{Q_s} = \]

(16)

It is clear that

\[ \min_{\psi \in \mathbb{P}_s, \phi \in \mathbb{P}_s} \max_{\psi \in \mathbb{P}_s, \phi \in \mathbb{P}_s} \| F(t_s, \tau_{i_*}) z_{i_*} + \sum_{s \in I} (G_s \psi_s + Q_s \phi_s) - x \|^2_{Q_s} \]

(17)

Then (16) follows from (12), (13), (17), (18).

Suppose that equalities (7) are proved for \( s = i + 1, \ldots, i_* \) with \( i \leq i_* - 1 \). Let us prove (7) for \( s = i \). On account of (5) and (8), for this purpose, one needs to show that

\[ \min_{u^{(i)} \in \mathbb{R}^{(i)}} \max_{w^{(i)} \in \mathbb{R}^{(i)}} \| F(t_s, \tau_i) z_i + \sum_{s \in I} F(t_s, t)(bu_s(t) + gw_s(t)) dt - x \|^2_{Q_i} = \]

(18)

and it follows from (11), (15) that

\[ \min_{\psi \in \mathbb{P}_i, \phi \in \mathbb{P}_i} \max_{\psi \in \mathbb{P}_i, \phi \in \mathbb{P}_i} \| F(t_s, \tau_i) z_i + \sum_{s \in I} (G_s \psi_s + Q_s \phi_s) - x \|^2_{Q_i} \]

(19)

with

\[ \bar{y}_{i+1}(z_i, u^{(i)}(), \psi^{(i)}()) = F(t_{i+1}, \tau_i) z_i + \sum_{s \in I} F(t_{i+1}, t)(bu_s(t) + gw_s(t)) dt, \]

(20)

It follows from (10), (14), (20) that

\[ \min_{u^{(i)} \in \mathbb{R}^{(i)}} \max_{w^{(i)} \in \mathbb{R}^{(i)}} \| F(t_s, \tau_i) z_i + \sum_{s \in I} F(t_{i+1}, t)(bu_s(t) + gw_s(t)) dt - x \|^2_{Q_i} = \]

(21)

These equalities and equalities (12), (13), and assumption that \( \bar{y}_{i+1}(x | z_{i+1}) = y_{i+1}(x | F(t_{i+1}, \tau_i) z_{i+1}) \) imply relations (19), and, consequently, relations (7) for \( s = i \). The theorem is proved.

**Corollary 1.** The sets \( X_*(z_i | \tau_{i_*}, \ldots, \tau_i) \), \( i = 0, \ldots, i_* \), can be presented in the form

\[ X_*(z_i | \tau_{i_*}, \ldots, \tau_i) = \{ x \in \mathbb{R}^n : \gamma_i(x | F(t_s, \tau_i) z_i) \leq \mu_i \}, \]

(21)

where functions \( \gamma_i(x | d), i = 0, \ldots, i_* \), are defined according to relations (8), (9).

It follows from presentation (21) that, for given \( z_i, \tau_i, \tau_{i+1}, \ldots, \tau_{i_*} \), the reachable set \( X_*(z_i | \tau_{i+1}, \ldots, \tau_{i_*}) \) is the level set for the function \( \gamma_i(x | F(t_s, \tau_i) z_i), x \in \mathbb{R}^n \).
Theorem 2. For $i = 0, \ldots, i_*$, function $\gamma_i(x|d)$, $x \in \mathbb{R}^n, d \in \mathbb{R}^n$, is convex with respect to (w.r.t.) the set of variables $(x,d)$ on $X \times D$ where $X, D$ are some convex bounded subsets of $\mathbb{R}^n$.

In proving the theorem, we will use the following propositions.

Proposition 2. (see page 48 in [10]) Consider the function
\[ f(x) = \sup_{y \in Y} \phi(x, y), \ x \in X, \]
where $X \subset \mathbb{R}^n$ is a convex set, the function $\phi(x, y)$ is convex w.r.t. $x$ on $X$ for any fixed $y \in Y$, the function $f(x)$ is finite on $X$. Then the function $f(x)$ is convex w.r.t. to $x$ on $X$.

Proposition 3. (see Lemma 4.4 in [10]) Consider the function
\[ f(x) = \inf_{y \in Y} \phi(x, y), \ x \in X, \]
where $X \subset \mathbb{R}^n$ is a convex set, the function $\phi(x, y)$, is convex w.r.t. $x$ on $X$ for any fixed $y \in Y$, the function $f(x)$ is finite on $X$. Then $f(x)$ is convex w.r.t. to $x$ on $X$.

Proposition 4. Let $X, Y, Z, D$ be any convex bounded subsets of $\mathbb{R}^n$ and $D = \{ \tau \in \mathbb{R}^n : \tau = y + z + d, y \in Y, z \in Z, d \in D \}$.

Suppose that the function $f(x|d)$ is convex w.r.t. the set of variables $(x,d)$ on $X \times D$. Then the function $f(x|y,z,d) = f(x|y + z + d)$ is convex w.r.t. the set of variables $(x,y,z,d)$ on $X \times Y \times Z \times D$.

Proof of Theorem 2. It follows for the constructions presented above that the functions $\gamma_i(x|d)$, $i = 0, \ldots, i_*$, can be written in the form
\[ \gamma_{i_0}(x|d) = \min_{z \in Z_{i_0}} \max_{y \in Y_{i_0}} \|d + z + y - x\|_Q^2, \]
\[ \gamma_i(x|d) = \min_{z \in Z_{i+1}} \max_{y \in Y_{i+1}} \gamma_{i+1}(x|d + y + z), \ i = i_0 - 1, i_0 - 2, \ldots, 0, \]
where the sets $Z_{i}(i), Z_{i}(i), \ i = 0, \ldots, i_*$ are defined in (11), (15) and are convex.

We will prove the theorem on the base of induction approach.

Owing to Proposition 2, the function $f_i(x,z,d) = \max_{z \in Z_{i+1}} f_{i+1}(x,z,d)$ is convex w.r.t. the set of variables $(x,z,d)$ on $X \times Z \times D$ where $X, Z, D$ are convex subsets of $\mathbb{R}^n$. Consequently, in view of Proposition 3, the function $\gamma_i(x|d) = \min_{z \in Z_{i}} f_i(x,z,d)$ is convex w.r.t. the set of variables $(x,d)$ on $X \times D$, where $X, D$ are convex subsets of $\mathbb{R}^n$.

Suppose that for some $i, 0 < i \leq i_*, \gamma_{i+1}(x|d)$ is convex w.r.t. the set of variables $(x,d)$ on $X \times D$, where $X, D$ are convex subsets of $\mathbb{R}^n$. Then according to Proposition 4 function $f_{i+1}(x,y,z,d) = \gamma_{i+1}(x|y + z + d)$ is convex w.r.t. the set of variables $(x,y,z,d)$ on $X \times Y \times Z \times D$ where $X, Y, Z, D$ are convex subsets of $\mathbb{R}^n$. Hence due to Proposition 2 the function $f_i(x,z,d) = \max_{y \in Y_{i+1}} \gamma_{i+1}(x,y,z,d)$ is convex w.r.t. the variables $(x,z,d)$ on $X \times Z \times D$. Taking into account this fact and Proposition 3, we conclude that the function $\gamma_i(x|d) = \min_{z \in Z_{i}} f_i(x,z,d)$ is convex w.r.t. the set of variables $(x,d)$ on $X \times D$. The theorem is proved.

Corollary 2. For $i = 0, \ldots, i_*$, the set $X_i(z_i | r_{i_1}, \ldots, r_{i_0}) \subset \mathbb{R}^n$ is convex (or empty).
4. Dependence of the Reachability Set on a Set of Correction Moments

Let us consider two sets of correction time instances

\[ \tau_1, \tau_2, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l, \]

and

\[ \tau_1, \tau_2, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l, \]

where \( \tau_k = t_m, \ m_k < m_s < m_{k+1}, \ k \in \{0, 1, N\} \). It follows from Theorem 1 that the sets

\[ X_* (z_0 \mid \tau_0 = 0, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

and

\[ X_* (z_0 \mid \tau_0 = 0, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

can be presented in the forms

\[ X_* (z_0 \mid 0, \tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

and

\[ X_* (z_0 \mid 0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

Here

\[ \hat{y}(x \mid z_0, \tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) = \min_{x, \in \mathbb{R}^n} \max_{s, \in \mathbb{N}} \min_{x, \in \mathbb{R}^n} \max_{s, \in \mathbb{N}} \min_{x, \in \mathbb{R}^n} \max_{s, \in \mathbb{N}} \| F(t_s, 0)z_0 + \sum_{s=1}^{N+1} (G_s \psi_s + Q_s \phi_s) - x \|_2 \]  

subject to \( \psi_s^T G_s \psi_s \leq r_s, \ \phi_s^T Q_s \phi_s \leq v_s, \ s = 1, \ldots, N + 1 \).

The function \( \hat{y}(x \mid z_0, \tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \) is obtained from the function

\[ \hat{y}(x \mid z_0, \tau_0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

if in (22) operations \( \min \max \) are replaced by

\[ \min \max \min \max \]

\[ \psi_{s, \in \mathbb{N}} \phi_{s, \in \mathbb{N}} \psi_{s, \in \mathbb{N}} \phi_{s, \in \mathbb{N}} \]

and

\[ \psi_{s, \in \mathbb{N}} \phi_{s, \in \mathbb{N}} \psi_{s, \in \mathbb{N}} \phi_{s, \in \mathbb{N}} \]

Taking into account the well-known max-min inequality

\[ \min \max \min \max \ f(\psi, \psi^*, \phi, \phi^*) \leq \min \max \min \max \ f(\psi, \psi^*, \phi, \phi^*), \]

we conclude that the following inclusion takes place

\[ X_* (z_0 \mid 0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \subset X_* (z_0 \mid 0, \tau_1, \ldots, \tau_k, \tau_{k+1}, \ldots, \tau_l) \]

Hence we have proved the following theorem that characterizes the dependence of the reachable set on a set of correction moments.

**Theorem 3.** Addition of a new moment to a set of correction moments leads to the extension of the set of system states that are \((\mu_*, Q_*\) – guaranteed reachable at the terminal moment \(t = t_*\) form a fixed initial position \((z_0, 0)\).

In the next section we illustrate the theoretical result on a numerical example.

5. Numerical Experiment

Consider system (1). Suppose that \( N = t_* = 1 \) and the sets of moments \( t_s, s = 0, \ldots, N + 1 \) and \( \tau_*, s = 0, \ldots, l_*, \) are the following

\[ t_0 = 0, \ t_1 = 1, \ t_2 = 2, \ \tau_0 = t_0, \ \tau_1 = t_1. \]  

The aim of this example is to show that the reachable set \( X_* (z_0 \mid \tau_0, \tau_1) \) (with one correction moment \( \tau_1 \)) is wider than the reachable set \( X_* (z_0 \mid \tau_0) \) (without correction moments). For the comparison of these sets, we construct the corresponding approximations. The \( X_* (z_0 \mid \tau_0) \) is approximated by a set \( \overline{X_*} (z_0 \mid \tau_0) \) such that \( \overline{X_*} (z_0 \mid \tau_0) \supset X_* (z_0 \mid \tau_0) \) and the set \( X_* (z_0 \mid \tau_0, \tau_1) \) is approximated by the set \( \overline{X_*} (z_0 \mid \tau_0, \tau_1) \) such that \( \overline{X_*} (z_0 \mid \tau_0, \tau_1) \subset X_* (z_0 \mid \tau_0, \tau_1) \). After that we will show that \( \overline{X_*} (z_0 \mid \tau_0, \tau_1) \supset \overline{X_*} (z_0 \mid \tau_0) \supset X_* (z_0 \mid \tau_0) \).

Rules for construction of the auxiliary sets \( \overline{X_*} (z_0 \mid \tau_0) \) and \( \overline{X_*} (z_0 \mid \tau_0, \tau_1) \) are presented below.
5.1. Construction of the Set $\overline{X}_s(z_0 | \tau_0)$

For data (24), the set $X_s(z_0 | \tau_0)$ can be written in the form

$$X_s(z_0 | \tau_0) = \{ x \in \mathbb{R}^n : \min_{\psi_1, \ldots, \psi_s} \max_{\xi, \phi} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + \phi - x \|_Q \leq \mu_s \},$$

subject to $\psi^T_s G_s \psi_s \leq r_s, s = 1, 2$,

where $\Phi_s = \{ y \in \mathbb{R}^n : y = Q_1 y_1 + Q_2 y_2, y_1^T Q_1 y_1, y_2^T Q_2 y_2 \leq \nu_s, s = 1, 2 \}$.

Consider the set

$$X_s(z_0 | \tau_0) = \{ x \in \mathbb{R}^n : \min_{\psi_1, \ldots, \psi_s} \max_{\xi, \phi} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + \phi - x \|_Q \leq \mu_s \},$$

subject to $\psi^T_s G_s \psi_s \leq r_s, s = 1, 2$,

where $\Phi^*_s$ is a polyhedron which is an external approximation for the set $\Phi_s$. Since $\Phi^*_s \subset \Phi_s$, we have

$$X_s(z_0 | \tau_0) \supset X_s(z_0 | \tau_0).$$

To construct the set $X_s(z_0 | \tau_0)$ one needs to calculate the maximum of convex function over the polyhedron $\Phi^*_s$. It is well-known that the maximum is reached at a vertex of the polyhedron $\Phi^*_s$. Let $y_j, j \in J$, be the vertices of the polyhedron $\Phi^*_s$. Then the set $X_s(z_0 | \tau_0)$ can be written in the equivalent form:

$$X_s(z_0 | \tau_0) = \{ x \in \mathbb{R}^n : \xi(x) \leq \mu_s \}$$

where $\xi(x)$ is the optimal value of the following problem

$$\xi(x) = \min_{\xi, \psi_1, \psi_2} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + y - x \|_Q \leq \xi, j \in J, \psi^T_s G_s \psi_s \leq r_s, s = 1, 2.$$  \hspace{1cm} (25)

The problem (25) is a convex programming problem and its solution can be found by standard methods.

5.2. Construction of the Set $\overline{X}_s(\tau_0, \tau_1)$

For data (24), the set $X_s(\tau_0, \tau_1)$ can be presented in the form

$$X_s(z_0 | \tau_0, \tau_1) = \{ x \in \mathbb{R}^n : \min_{\psi_1, \ldots, \psi_s} \max_{\xi, \phi} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + \phi - x \|_Q \leq \mu_s \},$$

subject to $\psi^T_s G_s \psi_s \leq r_s, s = 1, 2$,

where $\Phi^*_s = \{ \phi \in \mathbb{R}^n : \phi = Q_1 \phi_1 + Q_2 \phi_2 \leq \nu_s \}, s = 1, 2$.

Consider the set

$$X_s(z_0 | \tau_0, \tau_1) = \{ x \in \mathbb{R}^n : \min_{\psi_1, \ldots, \psi_s} \max_{\xi, \phi} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + \phi - x \|_Q \leq \mu_s \},$$

subject to $\psi^T_s G_s \psi_s \leq r_s, s = 1, 2$,

where $\Phi^*_s$ is a polyhedron that is an external approximation for the set $\Phi_s$, $s = 1, 2$. Denote by $y_1^i, i \in J_1$, and $y_2^j, j \in J_2$, the vertices of the polyhedrons $\Phi^*_s$ and $\Phi^*_s$ respectively.

Since $\Phi^*_s \supset \Phi_s$, $s = 1, 2$, then the inclusion $X_s(z_0 | \tau_0, \tau_1) \subset X_s(z_0 | \tau_0, \tau_1)$ takes place.

Reasoning by analogy with subsection 5.1, we get the following presentation of the set

$$X_s(z_0 | \tau_0, \tau_1) = \{ x \in \mathbb{R}^n : \xi(x) \leq \mu_s \}$$

where $\xi(x)$ is the optimal value of the problem

$$\xi(x) = \min_{\xi, \psi_1, \psi_2} \| F(t_0, 0)z_0 + G_1\psi_1 + G_2\psi_2 + y_1^i + y_2^j - x \|_Q \leq \xi, j \in J_1, i \in J_2,$$

$$\psi^T_s G_s \psi_s \leq r_s, j \in J_1. \hspace{1cm} (26)$$

The problem (26) is a convex programming problem. Its solution can be found by standard methods.
5.3. Example

Consider system (1) with the following initial data: \( n = 2, A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = [0,1], g = [1/2,3/10], t_0 = 0, t_1 = 1, t_2 = 2, m_1 = 1, r_1 = 2, r_2 = 2, v_1 = 3, v_2 = 4, z_0 = [-1,1], \mu = 10.025, Q_* = \text{diag}(1,1). \)

The set \( \Phi_* \) and its internal approximation \( \Phi_*^M \) are shown in Figure 1.

It is easy to check that the boundary \( \partial \Phi_* \) of the set \( \Phi_* \) is described as follows

\[
\partial \Phi_* = \{z(t) \in R : z(t) = [\alpha_1(t)Q_1 + \alpha_2(t)Q_2]c(t), c(t) = (\cos(t), \sin(t))^T, t \in [0,2\pi]\}
\]

with \( \alpha_s(t) = \frac{v_s}{c^T(t)Q_*c(t)}, s = 1,2. \)

Figures 2 and 3 show the sets \( \Phi_1^*, \Phi_2^* \) and their external approximations \( \Phi_1^M, \Phi_2^M \) respectively.

To construct external approximations, polyhedrons \( \Phi_1^{1M} \) and \( \Phi_2^{2M} \), we choose some points \( z(t_i^s), i = 1,\ldots,p, \) on the boundaries

\[
\partial \Phi_1^{1M} = \{z(t) \in R : z(t) = \alpha_1(t)Q_*c(t), c(t) = (\cos(t), \sin(t))^T, t \in [0,2\pi]\}, s = 1,2,
\]

of the sets \( \Phi_1^* \) and \( \Phi_2^* \) respectively. For \( s = 1,2 \) and \( i = 1,\ldots,p \), at a point \( z(t_i^s) \) we draw the straight line that is normal to the corresponding vectors \( c(t_i^s) \). Crossing points of these lines form the vertices of the polyhedrons \( \Phi_*^{1M}, s = 1,2. \)

Figure 4 shows the sets \( \overline{X}_*(z_0 | \tau_0) \) (more dark colour) and \( \overline{X}_*(z_0 | \tau_0, \tau_1) \).
Since \( \overline{X}_\ast(z_0 \mid \tau_0) \subset \overline{X}_\ast(z_0 \mid \tau_0, \tau_1) \), then \( X_\ast(z_0 \mid \tau_0) \subset X_\ast(z_0 \mid \tau_0, \tau_1) \).

This example shows that adding one correction moment may lead to a substantial extension of the set of \( (\mu, Q_\ast) \) – guaranteed reachable states of the system.

**Conclusions**

In the paper, we give a description of the reachable set for linear controlled system with uncertainties under assumption that control function may be corrected in some intermediate time instances.

To realize this description one needs to solve min-max optimisation problems with respect to a set of unknown functions. It is proved that these complicated optimisation problems can be reduced to simpler optimisation problems with respect to a set of unknown \( n \)-vectors. This allows us to give more constructive description of the reachable set.

It is proved that adding one new correction moment leads to an extension of the set of system states that are \( (\mu_\ast, Q_\ast) \) – guaranteed reachable at the terminal moment \( t_\ast \) from a given initial position \( (z_0, 0) \). Numerical example shows that this extension may be substantial. Some rules for construction external and internal approximations for the reachable sets are discussed.

**References**


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