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*Transport and Telecommunication Institute, Lomonosova 1, Riga, LV-1019, Latvia*

# ON A PROBLEM OF SPATIAL ARRANGEMENT OF SERVICE STATIONS

***Alexander Andronov, Andrey Kashurin***

*Riga Technical University*
*Kalku Str.1, Riga, LV-1658 Latvia*
*E-mail: Aleksandrs.Andronovs@rtu.lv, andrejs.kasurins@sp.gov.lv*

A problem of service station arrangement in spatial space is considered. A density function of serviced object location and a function that describes the corresponding loss are known. As criteria of arrangement is an average total loss. For the optimisation the gradient method is used. Numerical examples illustrate the suggested approach to setting the problem solution

***Keywords:*** *spatial arrangement, service stations, gradient method*

## 1. Introduction

Let us consider a real space *X* for that concrete *point* will be marked by ***x***, for plane it is two-dimensional vector (it is available to consider another dimension too). *A distance* $l(x,x^*)$ is determined for points ***x*** and ***x****, that satisfies usual conditional of distance axioms: $l(x,x)=0$, $l(x,x^*)\geq 0$, $l(x,x^*)\leq l(x,x')+l(x',x^*)$.

Some *objects* are arranged in the space (for example men, animals, stationers). Let us name as ***x***–*object,* the object that is at the point ***x***. The density of object arrangement is described by known density function *f*(***x***) ≥ 0, so

$$\int_{x\in X} f(x)x = 1.$$

Some *service stations* must be arranged in the space, their number is *k*. It is necessary to determine those coordinates $x^{(1)},x^{(2)},...,x^{(k)}$. If a ***x***–object is serviced by *i*–th station then corresponding loss is equal to $g_x\left(x^{(i)}\right)$, for example $g_x\left(x^{(i)}\right)=g(x-x^{(i)})$. Let us call $g_x(\circ)$ as *loss function* and suppose that it is a symmetry according to zero ( $g_x(x^{(i)})=g_x(-x^{(i)})$ ) and convex (down).

All amount of service for the ***x***–object is deviated between various service stations according to inverse proportion of the distances from the ***x***–object and the station. Most precisely, a part of ***x***–object service that belongs to the *i-th* station is

$$\delta_i(x)=\frac{\left(l\left(x,x^{(i)}\right)\right)^{-1}}{\sum_j\left(l\left(x,x^{(j)}\right)\right)^{-1}}. \tag{1}$$

Now a problem can be formulated as follows: to find coordinates $x^{(1)},x^{(2)},...,x^{(k)}$ of station arrangement that minimizes the total loss:

$$D\left(x^{(1)},x^{(2)},...,x^{(k)}\right)=\int\frac{1}{\sum_{i=1}^{k}\left(l\left(x,x^{(i)}\right)\right)^{-1}}\sum_{i=1}^{k}\left(l\left(x,x^{(i)}\right)\right)^{-1}g_x(x^{(i)})f(x)dx. \tag{2}$$

The article is organized in the following way. At first one-dimensional case of space *X* and the corresponding example is considered. Then we consider two-dimensional case. The article ends by some conclusion remarks. The Appendix contains the analytical investigation of the simplest one-dimensional case when *k* = 1 and density *f*(***x***) and loss function $g_x(\circ)$ are symmetric functions.

## 2. One-Dimensional Case

At first we consider a case when space $X$ is real axis $R$ = (- ∞, ∞). We will use a gradient method for the minimization of criterion (2). For that aim let us calculate a corresponding gradient. For a partial derivative of (2) we have the following expression:

$$\frac{\partial}{\partial x^{(j)}} D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) = \frac{\partial}{\partial x^{(j)}} \int \frac{1}{\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1}} \sum_{i=1}^{k} \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) f(x) dx =$$

$$= \int \frac{1}{\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1}} \left( \left( g_x(x^{(j)}) \left( -\left(l\left(x, x^{(i)}\right)\right)^{-2} \right) \frac{\partial}{\partial x^{(j)}} l\left(x, x^{(j)}\right) + \frac{1}{l\left(x, x^{(j)}\right)} \frac{\partial}{\partial x^{(j)}} g_x(x^{(j)}) \right) \right) f(x) dx -$$

$$- \int \left( \sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} \right)^{-2} \left( \left( -\left(l\left(x, x^{(j)}\right)\right)^{-2} \right) \frac{\partial}{\partial x^{(j)}} l\left(x, x^{(j)}\right) \sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) \right) f(x) dx. \quad (3)$$

Now we are able to rewrite the gradient of $D$ as

$$\nabla D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) = \left( \frac{\partial}{\partial x^{(1)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}), \ldots, \frac{\partial}{\partial x^{(k)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) \right)^T. \quad (4)$$

To accelerate a convergence of the gradient method we use a two-stage procedure. At the first stage we use component-wise (coordinate-wise) modification of the gradient method. It means that a sequence of cycles is preformed. Each cycle contains $k$ iterations. During the $i$-th iteration ($j$ = 1, 2, …, $k$) function (2) is minimized with respect to coordinate $x^{(j)}$, at the same time other coordinates do not change. For that minimization the gradient method with gradient (3) is used. The cycles end when the change of function (2) is mall. In the second stage we calibrate the obtained result by using the usual gradient method with gradient (4).

## 3. Example of One-Dimensional Case

Let density function be a mixture of normal distributions with means $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}$ and variances $\left(\sigma^{(1)}\right)^2, \left(\sigma^{(2)}\right)^2, \ldots, \left(\sigma^{(r)}\right)^2$, and weighted coefficients $p_1, p_2, \ldots, p_r$ $\left(p_1 + p_2 + \ldots + p_r = 1\right)$:

$$f(x) = \sum_{i=1}^{r} p_i \frac{1}{\sqrt{2\pi}\sigma^{(i)}} \exp\left( -\frac{1}{2} \left( \frac{x - \mu^{(i)}}{\sigma^{(i)}} \right)^2 \right), -\infty < x < \infty. \quad (5)$$

Further let us use the following distance function and loss function:

$$l(x, z) = |x - z|, \quad (6)$$

$$g(x, z) = (x - z)^2. \quad (7)$$

Then we have the following derivatives:

$$\frac{\partial}{\partial z} l(x, z) = \begin{cases} 1 & if\ x < z, \\ -1 & otherwise, \end{cases} \quad (8)$$

$$\frac{\partial}{\partial z} g(x, z) = -2(x - z). \quad (9)$$

Now we are able to use formula (3) for optimization.

Let us consider the following numerical data: $k = 4$, $r = 9$ and

$$p = (0.1 \quad 0.2 \quad 0.15 \quad 0.05 \quad 0.1 \quad 0.06 \quad 0.04 \quad 0.13 \quad 0.17)^T,$$

$$\mu = (0 \quad 3.15 \quad 0.05 \quad 4.05 \quad 6.1 \quad 7.06 \quad 7.74 \quad 8.13 \quad 10.17)^T,$$

$$\sigma = (0.2 \quad 1.2 \quad 1.15 \quad 1.05 \quad 0.71 \quad 2.06 \quad 1.74 \quad 2.13 \quad 1.17)^T.$$

The Figure 1 contains an according graphic of density function $f(x)$.

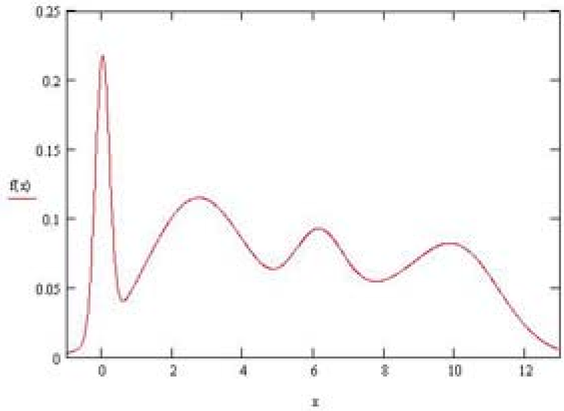


*Figure 1.* Plot of function f(x) for one-dimensional case

We begin the first stage of the optimization procedure with the values of coordinates $x = \left(x^{(1)}, x^{(2)}, .x^{(3)} x^{(4)}\right)^T = (1 \quad 2 \quad 4 \quad 7)^T$. It corresponds to $D = 9.766$ value of criterion (2). Table 1 contains the results of sequential cycles.

TABLE 1. Results of sequential cycles for one-dimensional case

| Iteration number | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x^{(1)}$ | 1 | 1 | 1 | 1 | 0.174 | 0.174 |
| $x^{(2)}$ | 2 | 2 | 2 | 3.259 | 3.259 | 3.259 |
| $x^{(3)}$ | 4 | 4 | 6.207 | 6.207 | 6.207 | 6.207 |
| $x^{(4)}$ | 7 | 9.809 | 9.809 | 9.809 | 9.809 | 10 |
| $D\left(x^{(1)}, x^{(2)}, .x^{(3)} x^{(4)}\right)$ | 9.766 | 9.327 | 8.818 | 7.727 | 7.494 | 7.468 |

We see that minimal value of criterion (2) is equal to $D = 7.468$ that is calculated by

$$x = (0.174 \quad 3.259 \quad 6.207 \quad 10)^T.$$

Further we perform the second stage of the optimization procedure and finally get minimal value $D = 7.445$ that corresponds to coordinates

$$x = (0.193 \quad 3.176 \quad 6.428 \quad 10.035)^T.$$

## 4. Two-Dimensional Case

Now we consider a case when space $X$ is real plane $R^2 = (-\infty, \infty) \times (-\infty, \infty)$. Then the coordinates of an object are $x = (x_1 \quad x_2)^T$, coordinates of the $j$-st station are $x^{(j)} = \left(x_1^{(j)} \quad x_2^{(j)}\right)^T$. Now instead of scalar derivative (3) we have two-dimensional vector

$$\frac{\partial}{\partial x^{(j)}} D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) = \left( \frac{\partial}{\partial x_1^{(j)}} D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) \quad \frac{\partial}{\partial x_2^{(j)}} D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) \right)^T .$$

Analogously to (3) we have for partial derivative ($i$ = 1, 2):

$$\frac{\partial}{\partial x_q^{(j)}} D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) = \frac{\partial}{\partial x_q^{(j)}} \int \frac{1}{\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1}} \sum_{i=1}^{k} \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) f(x) dx =$$

$$= \int \left( \sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} \right)^{-1} \left( \left( g_x(x^{(j)}) \left( -\left(l\left(x, x^{(j)}\right)\right)^{-2} \right) \frac{\partial}{\partial x_q^{(j)}} l\left(x, x^{(j)}\right) + \left(l\left(x, x^{(j)}\right)\right)^{-1} \frac{\partial}{\partial x_q^{(j)}} g_x(x^{(j)}) \right) \right) f(x) dx -$$

$$- \int \left( \sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} \right)^{-2} \left( \sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) \left( -\left(l\left(x, x^{(j)}\right)\right)^{-2} \right) \frac{\partial}{\partial x_q^{(j)}} l\left(x, x^{(j)}\right) \right) f(x) dx. \tag{10}$$

Now instead of (4) we have the ($2 \times k$)-matrix of the partial derivatives

$$\nabla D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) = \begin{pmatrix} \frac{\partial}{\partial x_1^{(1)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) & \ldots & \frac{\partial}{\partial x_1^{(k)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) \\ \frac{\partial}{\partial x_2^{(1)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) & \ldots & \frac{\partial}{\partial x_2^{(k)}} D(x^{(1)}, x^{(2)}, \ldots, x^{(k)}) \end{pmatrix}. \tag{11}$$

For the optimization we again use the two-stage procedure. At the first stage the component-wise (coordinate-wise) modification is used as follows. During the $j$-th iteration ($j$ = 1, 2, …, $k$) function (2) is minimized with respect to both coordinates of the $j$-st station $x^{(j)} = \left(x_1^{(j)} \quad x_2^{(j)}\right)$, at the same time other coordinates do not change. According to the gradient method we move along the gradient with respect to $\left(x_1^{(j)} \quad x_2^{(j)}\right)$, recalculating the one continually. At the second stage we work with the full gradient (11).

## 5. Example of Two-Dimensional Case

As before, let density function be a mixture of two-dimensional normal distributions with means $\mu^{(1)} = \left(\mu_1^{(1)} \quad \mu_2^{(1)}\right)^T, \ldots, \mu^{(r)} = \left(\mu_1^{(r)} \quad \mu_2^{(r)}\right)^T$ and variances $\sigma^{(1)} = \left(\sigma_1^{(1)} \quad \sigma_2^{(1)}\right)^T, \ldots, \sigma^{(r)T}, = \left(\sigma_1^{(r)} \quad \sigma_2^{(r)}\right)^T$ and weighted coefficients $p_1, p_2, \ldots, p_r \; (p_1 + p_2 + \ldots + p_r = 1)$:

$$f(x) = f\left(\left(x_1 \quad x_2\right)^T\right) = \sum_{i=1}^{r} p_i \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}\sigma_1^{(i)}\sigma_2^{(i)}} \times$$
$$\times \exp\left( -\frac{1}{2\left(1-\rho^2\right)} \left( \left( \frac{x_1 - \mu_1^{(i)}}{\sigma_1^{(i)}} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1^{(i)}}{\sigma_1^{(i)}} \right)\left( \frac{x_2 - \mu_2^{(i)}}{\sigma_2^{(i)}} \right) + \left( \frac{x_2 - \mu_2^{(i)}}{\sigma_2^{(i)}} \right)^2 \right) \right). \tag{12}$$

Further let us use the following distance function and loss function:

$$l(x, z) = \sqrt{\left(x_1 - z_1\right)^2 + \left(x_2 - z_2\right)^2} , \tag{13}$$

$$g(x, z) = \left|x_1 - z_1\right| + \left|x_2 - z_2\right| . \tag{14}$$

Then we have the following derivatives ($i$ = 1, 2):

$$\frac{\partial}{\partial z_q} l(x, z) = -\frac{1}{\sqrt{\left(x_1 - z_1\right)^2 + \left(x_2 - z_2\right)^2}} \left(x_q - z_q\right), \tag{15}$$

$$\frac{\partial}{\partial z_q} g(x,z) = \frac{\partial}{\partial z_q} g\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = \begin{cases} 1 \;\; if \;\; x_q < z_q, \\ -1 \;\; otherwise. \end{cases} \qquad (16)$$

Now we are able to use formula (3) for optimization.

Let us consider the following numerical data: $k = 4$, $r = 9$ and

$$p = \begin{pmatrix} 0.1 & 0.2 & 0.15 & 0.05 & 0.1 & 0.06 & 0.04 & 0.13 & 0.17 \end{pmatrix},$$

$$\mu = \begin{pmatrix} 0 & 2 & 3.15 & 4.05 & 6.1 & 7.06 & 7.74 & 8.13 & 10.2 \\ 3 & 2 & 1 & 8 & 3 & 6 & 3.1 & 6 & 7 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 0.2 & 1.2 & 1.15 & 0.7 & 2.1 & 1.76 & 1.74 & 2.13 & 1.2 \\ 1.3 & 0.4 & 1 & 0.8 & 1.3 & 1.7 & 1.5 & 1.6 & 2.7 \end{pmatrix},$$

$$\rho = \begin{pmatrix} 0 & 0.2 & -0.15 & -0.7 & 0.6 & 0.06 & -0.74 & 0.13 & -0.17 \end{pmatrix}.$$

The Figure 2 contains an according graphic of density function $f(x)$.

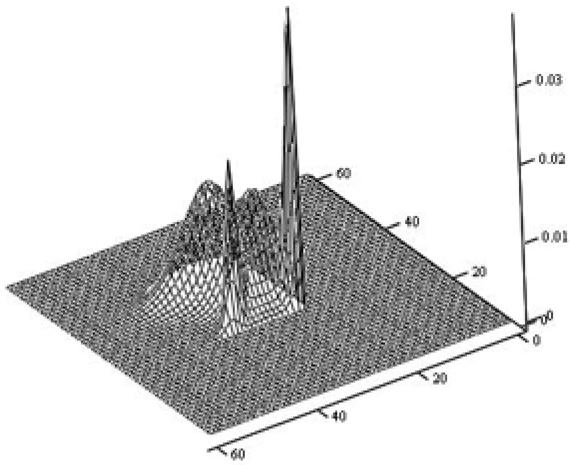


*Figure 2.* Plot of function f(x) for two-dimensional case

Table 2 contains the results of sequential cycles of the optimization procedure.

TABLE 2. Results of sequential cycles for two-dimensional case

| Iteration number | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_1^{(1)}$ | 0 | 0.961 | 1.099 | 1.134 | 1.212 | 1.190 |
| $x_2^{(1)}$ | 0 | 1.545 | 2.240 | 1.874 | 2.072 | 1.945 |
| $x_1^{(2)}$ | 3 | 2.703 | 2.944 | 3.130 | 3.099 | 3.079 |
| $x_2^{(2)}$ | 3 | 0.845 | 1.444 | 2.221 | 1.813 | 2.085 |
| $x_1^{(3)}$ | 6 | 6.115 | 6.170 | 6.303 | 6.361 | 6.487 |
| $x_2^{(3)}$ | 6 | 5.398 | 5.206 | 4.889 | 4.737 | 4.404 |
| $x_1^{(4)}$ | 8 | 8.141 | 8.194 | 8.303 | 8.335 | 8.381 |
| $x_2^{(4)}$ | 8 | 7.251 | 7.053 | 6.742 | 6.640 | 6.466 |
| $D$ | 5.173 | 4.616 | 4.391 | 4.328 | 4.304 | 4.296 |

From the Table we can see how the gradient method improves the criterion of value continually.

## Conclusion

A problem of service station arrangement in spatial space is considered. The elaborated algorithm of the problem solution is based on the gradient method. The considered numerical examples show its efficiency. The authors intend to apply the suggested approach to solving the practical arrangement problems.

## APPENDIX

Now we will consider the simplest one-dimensional case when $k = 1$ (one station only) and density $f(x)$ and loss function $g_x(\circ)$ are symmetric functions. Let $f(x)$ have a maximal value at the symmetry point $x = m$ and $g_x\left(x^{(1)}\right) = g(x - x^{(1)})$ have minimal value $g^*$ at the symmetry point 0. Now instead of (2) we have the following criteria:

$$D\left(x^{(1)}\right) = \int_{-\infty}^{\infty} g(x - x^{(1)}) f(x) dx = \int_{-\infty}^{\infty} g(m + x - x^{(1)}) f(m + x) dx. \qquad (17)$$

As $f$ is a symmetric function respectively $m$, $f(m + x) = f(m - x)$, then for the sum of two points $m + x$ and $m - x$, we have the following sum of the integral expression in (17):

$$g(m + x - x^{(1)}) f(m + x) + g(m - x - x^{(1)}) f(m - x) =$$
$$= f(m + x)\left(g(m + x - x^{(1)}) + g(m - x - x^{(1)})\right).$$

The convexity of function $g$ gives us

$$g(m + x - x^{(1)}) + g(m - x - x^{(1)}) = 2\left(\frac{1}{2} g(m + x - x^{(1)}) + \frac{1}{2} g(m - x - x^{(1)})\right) \geq$$
$$\geq 2g\left(\frac{1}{2}\left(m + x - x^{(1)}\right) + \frac{1}{2}\left(m - x - x^{(1)}\right)\right) = 2g(m - x^{(1)}) \geq 2z.$$

The lower limit is obtained if $x^{(1)} = m$. Therefore,

$$D\left(x^{(1)}\right) = \int_{-\infty}^{\infty} g(x - x^{(1)}) f(x) dx = \int_{0}^{\infty} g(m + x - x^{(1)}) f(m + x) dx + \int_{0}^{\infty} g(m - x - x^{(1)}) f(m - x) dx =$$
$$= \int_{0}^{\infty} f(m + x)\left(g(m + x - x^{(1)}) + g(m - x - x^{(1)})\right) dx \geq z,$$

which is obviously clear.

Taking derivative with respect to $x^{(1)}$ and equate one to zero, we get

$$\frac{\partial}{\partial x^{(1)}} D\left(x^{(1)}\right) = -\int \frac{\partial}{\partial x^{(1)}} g(x - x^{(1)}) f(x) dx = \int \frac{\partial}{\partial x^{(1)}} g(m + x - x^{(1)}) f(m + x) dx = 0 .$$

As function $g$ has the minimum at the point 0, and then the derivative from $g(x)$ is negative for $x < 0$ and is positive for $x > 0$. Therefore,

$$\int_{-\infty}^{x^{(1)} - m} \left| \frac{\partial}{\partial x^{(1)}} g(m + x - x^{(1)}) \right| f(m + x) dx = \int_{x(1) - m}^{\infty} \frac{\partial}{\partial x^{(1)}} g(m + x - x^{(1)}) f(m + x) dx.$$

Obviously, the unique solution is $x^{(1)} = m$. Therefore, for optimal value we have the following expression:

$$D(m) = \int_{-\infty}^{\infty} g(x-m) f(x) dx = \int_{-\infty}^{\infty} g(x) f(m+x) dx.$$

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