INSPECTION PROGRAM DEVELOPMENT USING MINIMAX METHOD

Maris Hauka, Yuri Paramonov

Aviation Institute of Riga Technical University
Kalku str. 1, LV-1658, Riga, Latvia
E-mail: maris.hauka@gmail.com, yuri.paramonov@gmail.com

The paper describes development of the methodology of the choice of aircraft fleet's inspection intervals. This methodology is verified, using the experimental fatigue crack growth data. Methodology includes in the economic parameters. Using this method it is possible to find safe- enough and the most cost- effective inspection planning programs. In addition the paper includes testing of the methodology at different permissive withdrawal probabilities and their impact on the real probability of withdrawal and the number of inspection intervals for the model with one aircraft and with aircraft fleet. For the case when they are not known the minimax approach is offered if there are results of acceptance full-scale fatigue test and conditions of acceptance are known. Numerical examples are given.

Keywords: Engineering inspection program, Markov chains, reliability

1. Introduction

This paper deals with the problem of the reliability of airlines and it is a kind of review and development of previous publications [1-6] where the problem of the reliability of fatigue-prone aircraft has been considered. The reliability of aircraft (AC) and airline (AL) operations can be ensured by the implementation of a specific inspection program, which can be planned using full-scale fatigue test data and the theory of Markov Chains (MC) and a Semi-Markov process (SMP). The process of the operation of AC is considered as absorbing MC with $(n+4)$ states. The states $E_1, E_2, ..., E_{n+1}$ correspond to AC operation in time intervals $[t_0, t_1), [t_1, t_2), ..., [t_n, t_{SL})$, where n is an inspection number, t_{SL} is specified life (SL), i. e. AC retirement time. States E_{n+2}, E_{n+3}, and E_{n+4} are absorbing states: AC is withdrawn from service when the SL is reached, fatigue failure (FF) and fatigue crack detection (CD). In the corresponding matrix for the operation process of AL the states E_{n+2}, E_{n+3} and E_{n+4} are not absorbing , but correspond to the return of the MC to state E_1 (AL operation returns to first interval). In the matrix of transition probabilities of AC , P_{AC}, there are three units in the three last lines in the diagonal, but for corresponding lines in the matrix for AL, P_{AL}, the units are in the first column, corresponding to state E_1. Using P_{AC} we can get the probability of FF of AC and cumulative distribution function, mean and variance of AC life, and the same characteristics under the condition of absorption in a specific absorbing state. Using P_{AL} we can get the stationary probabilities of AL operation $\{\pi_1, ..., \pi_{n+1}, \pi_{n+2}, ..., \pi_{n+4}\}$. Here π_{n+3} defines the part of MC steps when FF takes place and the MC appears in state E_{n+3}. The ratio of this value to the mean life of a new aircraft defines the intensity of FF, λ_F, i.e. the number of FF in one time unit. It can be calculated also using the theory of Semi-Markov Process (SMP) with rewards [7,8]. Using this theory we can calculate also the gain of the process. The problem of inspection planning is the choice of the sequence $\{t_1, t_2, ..., t_n, t_{SL}\}$ (in the case of equal inspection intervals and fixed t_{SL}, choice of n) corresponding to the maximum gain, taking into account the limitations imposed by the intensity of AL fatigue failure of AL or AC fatigue failure. A numerical example is given.

2. Formal Setting of the Problem

For the formal setting of the problem we should define the matrix of transition probabilities of MC and the matrix of rewards for SMP. However, first the fatigue crack growth model should be defined.

2.1. Fatigue crack growth model

We suppose the following exponential approximation of fatigue crack growth function when the size of a fatigue crack is defined by the equation:

$$a_{cr}(t) = a_{cr}(0)\exp(Qt).\tag{1}$$

Let T_d represents the time when a crack reaches detectable size, a_d. It is the time when the probability of crack discovery is equal to a unit if inspection is made (we suppose that before T_d this probability is equal to zero), T_c is the time when the crack reaches its critical size, a_c, and FF takes place. The value of a_c corresponds to the minimum residual strength of an aircraft component allowed by special design regulations. We have

$$T_d = (\ln a_d - \ln a_0)/Q = C_d/Q,\tag{2}$$

$$T_c = (\ln a_c - \ln a_0)/Q = C_c/Q,\tag{3}$$

where a_0 is a(0). Despite its simplicity, the formula (1) in interval $[T_d, T_c)$ shows clear results.

The values Cc, and Cd, can be derived each from another.

$$C_d = C_c - \delta,\tag{4}$$

$$C_c = C_d + \delta,\tag{5}$$

$$\delta = \ln a_c - \ln a_d = \ln \frac{a_c}{a_d}.\tag{6}$$

If δ and a_0 are constants then for considering the model of fatigue crack growth, the distribution of random variables (r.v.) T_d, T_c is defined by distribution of only two r.v.: Cc and Q.

Let us denote

$$X = \ln Q, Y = \ln C_c = \ln\left(\ln\left(a_c/a_d\right)\right).\tag{7}$$

From the analysis of the fatigue test data it can be assumed, that r.v. $\ln T_c$, is distributed normally. It comes from the additive property of the normal distribution that $\ln T_c$ could be normally distributed if both $\ln C_c$ and $\ln Q$ are normally distributed or if one of these values is constant but the other has normal distribution. Here we consider the simplest case. We suppose that random variable X has normal distribution with unknown mean value θ_0 and known θ_1 but Y is constant.

2.2. Failure of AC probability calculation using the theory of absorbing Markov chains

Let the probability of crack detection during the inspection number i at time moment t_i be denoted as v_i; the probability of failure in service time interval (t_{i-1}, t_i) as q_i; and the probability of absence of the above-mentioned events as u_i. These three events form a complete set $u_i + v_i + q_i = 1$. In our model we also assume that there is an inspection at t_{SL}. This inspection at the end of (n+1)-th interval does not change the reliability, but it should be done in order to know the state of aircraft (to know if there is any FC or if there are no fatigue cracks). *Effective number of inspections* is equal to n but in following, for short, we say that n is *number of inspections*. The transition probability matrix of this process, P_{AC}, can be composed as it is presented on Figure 1.

	E_1	E_2	E_3	...	E_{n-1}	E_n	E_{n+1}	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1	0	u_1	0	...	0	0	0	0	q_1	v_1
E_2	0	0	u_2	...	0	0	0	0	q_2	v_2
E_3	0	0	0	...	0	0	0	0	q_3	v_3
...	...	...	...	...	...	...	...	...	...	...
E_{n-1}	0	0	0	...	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
E_n	0	0	0	...	0	0	u_n	0	q_n	v_n
E_{n+1}	0	0	0	...	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
E_{n+2} (SL)	0	0	0	...	0	0	0	1	0	0
E_{n+3} (FF)	0	0	0	...	0	0	0	0	1	0
E_{n+4} (CD)	0	0	0	...	0	0	0	0	0	1

Figure 1. The transition probability matrix for absorbing MC

An example of a corresponding state transition diagram for the case of three inspections is shown on Figure 2. We suppose that if ,in interval $[T_d , T_c)$, some inspection is made, then fatigue failure will be eliminated.

$$u_i = P(T_d > t_i \,|\, T_d > t_{i-1}) = P(Q < C_d / t_i) / P(Q < C_d / t_{i-1}); \tag{8}$$

$$q_i = P(t_{i-1} < T_d < T_c < t_i \,|\, Td > t_{i-1}) =$$
$$= \begin{cases} 0, & if \quad t_{i-1}C_c / C_d > t_i, \\ \dfrac{P(C_c / t_i < Q < C_d / t_{i-1})}{P(Q < C_d / t_{i-1})}, & if \quad t_{i-1}C_c / C_d \le t_i, \end{cases} \tag{9}$$

$$v_i = 1 - u_i - q_i \tag{10}$$

Unfortunately, in general the corresponding integrals are not expressed by elementary functions, but if it is assumed that *Cc* and *Cd* are some constants, then

$$u_i = a_i / a_{i-1} , \tag{11}$$

$$q_i = \max(0, (a_{i-1} - b_i)/(1 - a_{i-1})) , \tag{12}$$

$$a_i = \Phi\left(\ln(C_d / t_i) - \theta_0 \right)/ \theta_1), \tag{13}$$

$$b_i = \Phi\left(\ln(C_c / t_i) - \theta_0 \right)/ \theta_1), \tag{14}$$

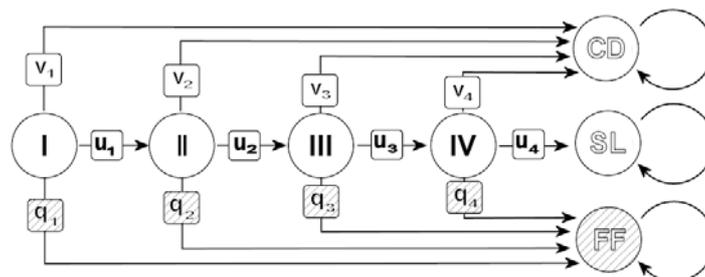*Figure 2.* State transition diagram for a three-inspection program

$\Phi(.)$ is a distribution function of a standard normal variable.

The structure of considered matrices can be described as follows on Figure 3.
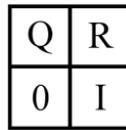

Figure 3. Sub-matrices of transition probabilities matrix

Here I is a matrix of identity corresponding to absorbing states, 0 is a matrix of zeros. Then the matrix of probabilities of absorbing in different absorbing states for different initial transient states is defined by formula (15)

$$B = (I - Q)^{-1} R.$$ (15)

The first row of the matrix B defines the probabilities of absorption in states SL, FF, CD if the initial state is E_1. Particularly, $B(1,2)$ defines the failure probability of a new aircraft which begins operation during the first interval. The following rows of the matrix B define the same probabilities for different initial states (for aircraft which begin operation in different time intervals). Therefore, if, for example, aircraft begin operation in the first interval, the failure probability of aircraft in the fleet is defined by equation (16)

$$p_f = aBb,$$ (16)

where vector row $a = (1,0,...,0)$ means that all aircraft begin operation in the first interval (state E1), vector column $b = (0,1,0)$ '.

3. Calculation of the Cost of Airline Operation

In order to analyze the cost of permanent operations of AL or aircraft fleet, it is necessary to modify the transition probability matrix.

Now the process is not absorbed or stopped in states SL, FF or CD. If these states are reached, the process restarts from the initial state E_1. This means that an entirely new aircraft is acquired. The modified transition probability matrix, P_{AL}, is shown on Figure 4. An example of the state transition diagram is shown on Figure 5.

	E_1	E_2	E_3	...	E_{n-1}	E_n	E_{n+1}	E_{n+2} (SL)	E_{n+3} (FF)	E_{n+4} (CD)
E_1	0	u_1	0	...	0	0	0	0	q_1	v_1
E_2	0	0	u_2	...	0	0	0	0	q_2	v_2
E_3	0	0	0	...	0	0	0	0	q_3	v_3
...	...	...	...	...	...	...	...	...	...	...
E_{n-1}	0	0	0	...	0	u_{n-1}	0	0	q_{n-1}	v_{n-1}
E_n	0	0	0	...	0	0	u_n	0	q_n	v_n
E_{n+1}	0	0	0	...	0	0	0	u_{n+1}	q_{n+1}	v_{n+1}
E_{n+2} (SL)	1	0	0	...	0	0	0	0	0	0
E_{n+3} (FF)	1	0	0	...	0	0	0	0	0	0
E_{n+4} (CD)	1	0	0	...	0	0	0	0	0	0

Figure 4. The transition probability matrix for stationary AL operation process
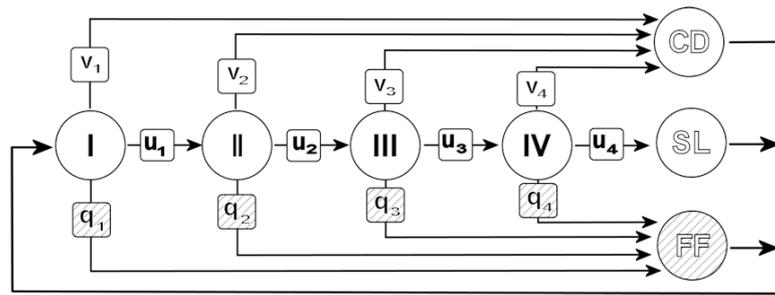
Figure 5. State transition diagram for stationary AL operation process with three inspections

Next we will consider economic analysis. The theory of Semi-Markov process with rewards is usually used to find the solution of similar problems [8]. A reward structure is described by the reward matrix R, the component of which, r_{ij}, describes the reward, connected with the transition from state E_i to state E_j; here $i,j=1,2,\ldots,n+4$. Let us define the reward, related to successful transitions from one operation interval to the following one by value $a(n)$; the reward related to transitions to state CD (or E_{n+4}) from any state $E_1, E_2, \ldots, E_{n+1}$ – by value b, to state FF (or E_{n+3}) – by value c; and from states SL, FF, CD to state E_1 (acquisition of new AC) – by value d.

Let us define the airline gain:

$$g(n) = \sum_{i=1}^{n+4} \pi_i g_i(n), \tag{17}$$

where $\pi = (\pi_1, \ldots, \pi_{n+4})$ is the vector of stationary probabilities, which is defined by the equation system

$$\pi P = \pi, \quad \sum_{i=1}^{n+4} \pi_i = 1; \tag{18}$$

$$g_i(n) = \begin{cases} a(n) \cdot u_i + b \cdot q_i + c \cdot v_i, & i = 1, \ldots n+1, \\ d, & i = n+2, \ldots, n+4 \end{cases}; \tag{19}$$

$u_i, q_i, v_i, \ i = 1, \ldots, n+1$, are probabilities of successful transitions from one to the following interval, to E_{n+3} and E_{n+4} states;

$a(n) = a_0(n) + d_{insp} t_{SL}$, where $a_0(n) = a_1 t_{SL} / (n+1)$, – is the reward, related to successful transitions from one operation interval to the following one, and the cost of one inspection (negative value), $d_{insp} t_{SL}$, which is supposed to be proportional to t_{SL} if it is supposed that all intervals are equal, a_1 defines the reward of operation in one time unit (one hour or one flight). The dimension of $a(n)$ should coincide with dimensions of t_{SL});

$b = b_1 a_0(n)$, $c = c_1 a_0(n)$ are the rewards related to transitions from any state $E_1, \ldots, E_{n+1}$ to state E_{n+3} (FF takes place) and E_{n+4} (CD takes place) which are supposed to be proportional to a_0; $d = d_1 t_{SL}$ is the negative reward, the value of which is the cost of new aircraft after events SL, FF or CD and transition to E_1 take place (it is supposed to be proportional to t_{SL}).

If $a(n) = b = c = 1$, $d = 0$ and time transition to state E_1 are equal to zero, then $\pi_{tj} = \pi_j g_j(n) / g(n)$ defines the part of time which SMP spends in state $E_j, j = 1, \ldots, n+1$; $L_j = g(n) / \pi_j$ defines the mean return time for state E_j; specifically, L_1 is the mean time of renewal of

AL operation in the first interval, L_{n+3} is the mean time between FF; so $\lambda_F = 1 / L_{n+3}$ is the intensity of fatigue failure. It should be remembered that the same value can be obtained using the theory of absorbing MC. This value is equal also to the ratio of aircraft failure probability to the mean life of new aircraft.

The problem is to maximize the gain, $g(n)$, under limitations of probability of aircraft FF or AL intensity of fatigue failure λ_F. In the following numerical example we consider the last version of the problem.

4. Choice of Inspection Number If Parameter of Fatigue Crack Growth Trajectory Is Known

Usually in Aircraft Design Bureau documents the sequence $(t_1,...,t_n)$ is defined by equation $t_i = t_1 + (i-1)(t_{SL} - t_1) / n$, $i = 1, 2,..., n+1$, with specific choice of t_1. Then we should choose only t_1 and n. To simplify the numerical example we suppose that all intervals of operation are equal. On Figures 6 and 7 we see numerical examples of calculations of gain, g, and of intensity of fatigue failure, λ_F, as functions of n.

The procedure of choice of n is the following. Firstly, we should calculate $g(n)$ and choose the number of n, corresponding to the maximum of gain g: $n_g = \arg\max g(n)$. Then, we should calculate expected value of intensity of fatigue failure, $\lambda_F(n)$, which is a function of n, and choose n_λ – it is such value of n that for any large inspection number the function $\lambda_F(n)$ will be equal or less than λ_F^*, which the value intensity of fatigue failure, allowed by specific aviation regulation

$$n_\lambda = \min\left\{ n : \lambda_F(n) \le \lambda_F^*, \ \text{for all } n \ge n_\lambda \right\}.$$

And finally, we should choose $n = \max(n_g, n_\lambda)$.

Consider the numerical example of calculation of gain, g, and intensity of fatigue failure, λ_F, for $n=2$ and following initial data: $t_{SL} = 40\,000$, $\theta_0 = -8.5885$, $\theta_1 = 0{,}34600$, $a_0 = 0.2181$; $a_c = 102$, $a_d = 20$, (these values have been used already, for example, in [1]); $a_1 = 1$, $b_1 = 0.05$, $c_1 = -0.05$, $d_1 = -0.3$, $d_{insp} = -0.0001$.

Using equation (10-14) we have (using the MATLAB notation for the matrix and vector)

$P_{AC} =$

[0	9.40e-001	0	0	6.38e-003	5.40e-002
0	0	3.47e-001	0	2.69e-001	3.85e-001
0	0	0	1.60e-001	2.42e-001	5.97e-001
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1]

In matrix P_{AL} the first three lines are the same, but in the last three lines the units are in the first column.

Using P_{AL} and equations (18) we have the stationary distribution for the case when time unit is one step in MC: $\pi = [3.06e-001, 2.88e-001, 9.97e-002, 1.60e-002, 1.04e-001, 1.87e-001]$.

Let us define the matrix P_{uqv} with u_i, q_i, v_i in every line for $i = 1,..., n+1$. In the example

$P_{uqv} =$
[9.3958e-001	6.3833e-003	5.4037e-002
3.4655e-001	2.6900e-001	3.8445e-001
1.6030e-001	2.4240e-001	5.9730e-001].

Then $g = \pi[P_{uqv}[a;b;c];d;d;d]$. In considered case $g = 1\,896$.

If the time dimension unit is the inspection interval then the vector of return times for every state

$$L = [g_t / \pi_1, ..., g_t / \pi_6] = [\,2.27,\ 2.41,\ 6.96,\ 4.34, 6.70,\ 3.72\,].$$

Here $g_t = \pi[1;1;1;0;0;0]$ is the mean airline gain if income is the time and the time unit is inspection interval. It is useful to note that $g_t = g$ if $a = b = c = 1$, $d = 0$. In considered case $g_t = 0.694$.

The return time of failure (to state FF), L_F, is equal to corresponding $(n+3)$-th component of vector $L: L_F = L(n+3) = 6.7$. The time length of one interval $t_1 = t_{SL} / (n+1) = 13\,333$.

The intensity of failure in t-time unit (flight or flight hour) $\lambda_F = (1 / L_F) / t_1 = 0.0000112$.

It is useful to note that the same values can be obtained using the theory of absorbing MC. We calculate the matrix of absorption

$$\begin{aligned} B = [&5.2196\text{e-}002 \quad 3.3806\text{e-}001 \quad\quad 6.0975\text{e-}001 \\ &5.5552\text{e-}002 \quad 3.5300\text{e-}001 \quad\quad 5.9145\text{e-}001 \\ &1.6030\text{e-}001 \quad 2.4240\text{e-}001 \quad\quad 5.9730\text{e-}001]. \end{aligned}$$

Then we take from it the probability of absorption of new aircraft in state FF, $p_f = 0.338$, calculate the mean time to absorption of new aircraft, $T_1 = \tau_1 t_1$ (where $\tau_1 = 2.27$ is the mean time to absorption if the time unit is inspection interval; its value coincides with $L(1)$!) and calculate the ratio $p_f / T_1 = 0.0000112$.

We see on Figures 8 and 9 results of similar calculations for $n=1,...,9$ after specific choice of designed failure intensity $\lambda_F = \lambda_{FD}$. Using this calculation we can make the choice of n.

5. Minimax Choice of Inspection Number

For the considered strategy for the choice of inspection number the real intensity of failure is a function of θ and designed allowed value of intensity of fatigue failure, λ_{FD}, which, as it will be shown, should not be equal to λ_F^*. But really we do not know θ and we can only, using test results, to get some estimate of this parameter, $\hat{\theta}$. So real intensity will be a random variable, $\lambda_F(\hat{\theta}, \lambda_{FD})$. We can limit the mean value of this function if we take into account that really full-scale fatigue test is approval test and redesign of airframe will be made (service of aircraft of tested design version will not be allowed) if some requirement are not met (for example, if it appears, that estimate of mean time to failure, $\hat{t}_C = C_C / \exp(\hat{\theta})$, is too small or calculated required inspection number, based on using $\hat{\theta}$ instead of θ is too great. If full-scale fatigue test is approved and there is not necessity to make airframe redesign let us write $\hat{\theta} \in \Theta_0$, where Θ_0 is some part of parameter space. Let us denote corresponding fatigue intensity function by

$$\lambda_F(\hat{\theta}, \lambda_{FD}, \Theta_0) = \begin{cases} \lambda_F(\hat{\theta}, \lambda_{FD}) & \text{if } \hat{\theta} \in \Theta_0, \\ 0 & \text{if } \hat{\theta} \notin \Theta_0 \end{cases} \tag{20}$$

On Figure 6 we see example of calculation of expected value of this function for different θ_0. Additionally, we define, that there is event $\hat{\theta} \in \Theta_0$ if $mean\_t_C = C_C / \exp(\theta_0) \geq k_R t_{SL}$. For Fig. 8 $k_R = 1$, $\lambda_{FD} = 0.000001$.
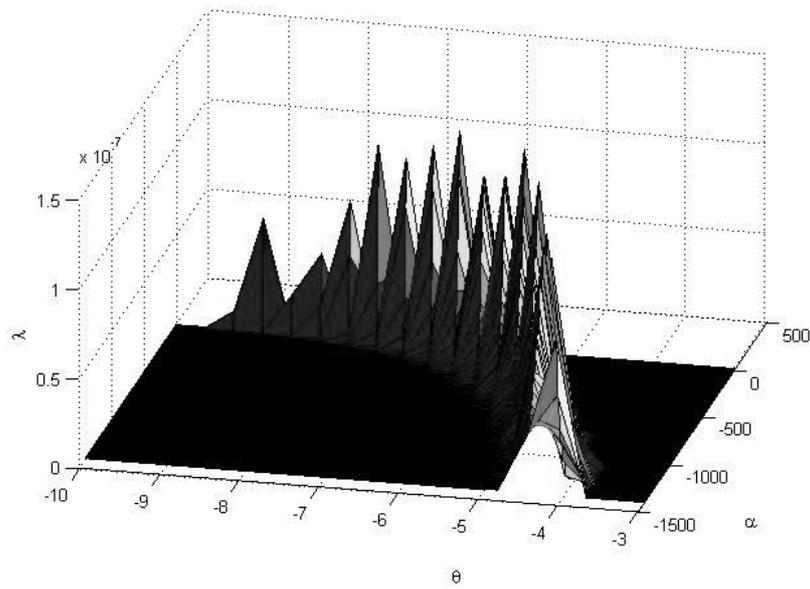
Figure 6. Mean intensity of fatigue failure as function of mean_t_C
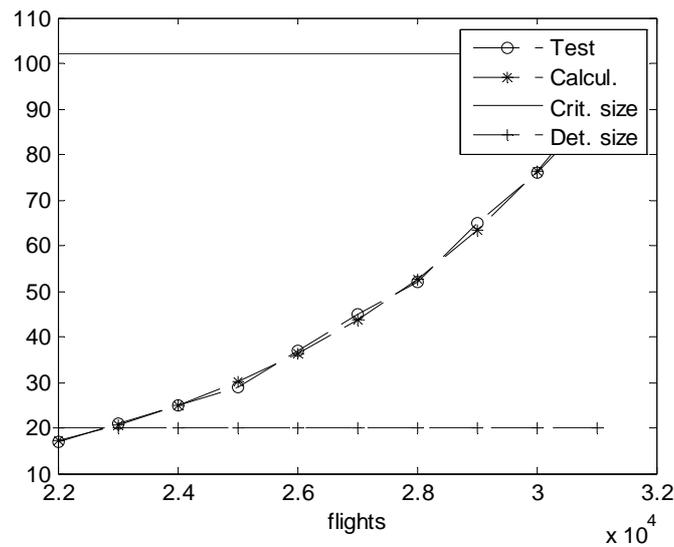

Figure 7. Example of fatigue crack size as function of flight number (result of full scale fatigue test)

We see that there is maximum value of λ_F because if design is too bad and test is not approved, then we redesign the airframe. But if it is too good then fatigue failure does not take place before the specified life will be reached. Maximum value of λ_F is equal to $1.6*10^{-6}$, it is in 1.6 time more than designed λ_{FD} =0.000001. So we can use this value of λ_{FD} for calculation of inspection number only if allowed failure intensity $\lambda^* \geq 0.0000016$. Suppose that just it is required reliability.

On Figure 7 we see example of fatigue crack, which was observed in full-scale fatigue test of some airframe. Using this crack data and regression analysis we can get estimates $\hat{\theta} = \ln Q$ =-8.5885. For these data we can make calculations of estimates of inspection intensity λ_F and gain g as function of inspection number and, finally, can find required inspection number (see Fig. 8). In this example $n_L = 6$; n_g =3; $n = \max(n_g, n_\lambda) = 6$
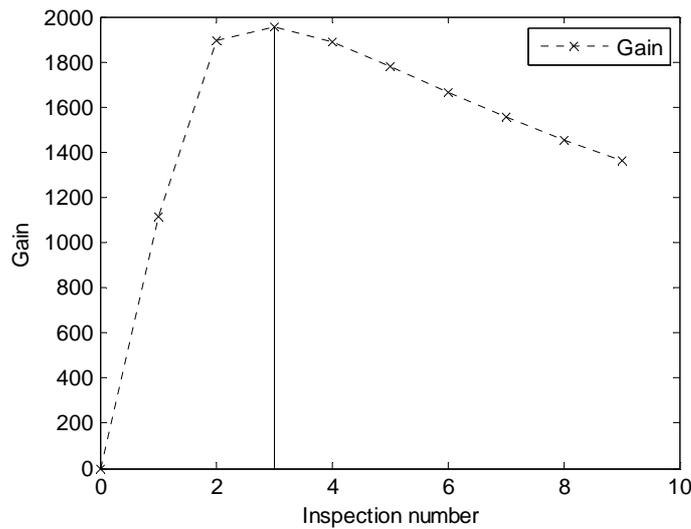
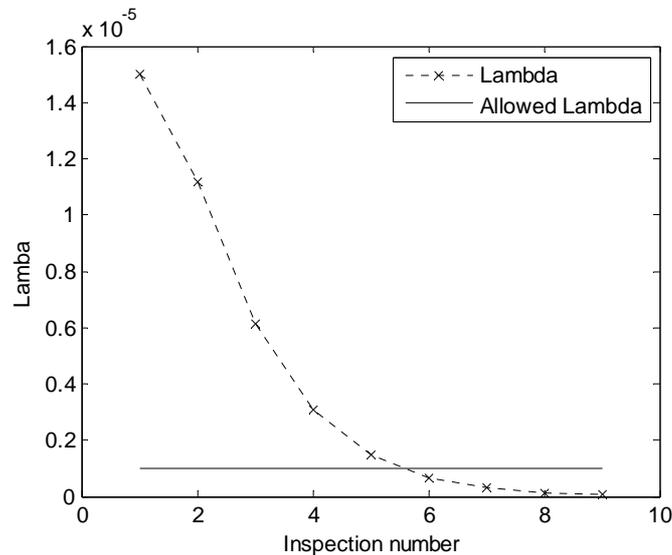*Figure 8.* Gain as function of inspection number

Figure 9. Intensity of failure as function of inspection number

6. Conclusions

This paper presents, (based on MC and SMP with rewards theory), a method to solve the problem of inspection planning corresponding to maximum airline gain while airline fatigue failure intensity is limited.. If the parameters of fatigue crack growth exponential models are unknown, but there are results of approval full-scale fatigue test of corresponding airframe the minimax approach (see [1]) should be used. Numerical examples are given.

References

1. Paramonov, Yu., Kuznetsov, A. Reliability of Fatigue-Prone Airframe, *International Review of Aerospace Engineering (IRAEASE)*, Vol. 2, 2009, pp. 145-153,
2. Paramonov, Yu., Kuznetsov, A. Using of p-set function for airframe inspection program development, *Communication of Dependability and Quality Management*, Vol. 9, 2006, pp. 51-55.

3. Paramonov, Yu.M. Aircraft fatigue problem solution by use of modern mathematical statistics methods, *Aviation*, No 6, 2002, pp. 83-96. /Vilnius: Technika/

4. Paramonov, Yu.M. P-set function application for specified life nomination and inspection program development. In: *Probabilistic Analysis of Rare Events: Theory and Problems of Safety. Insurance and Ruin. Proceedings of international conference. Jurmala, Latvia. June 28 - July 3,* Riga: RAU, 1999, pp. 202-208.

5. Paramonov, Yu.M. and A.Yu. Paramonova. Inspection program development by the use of approval test results, *International Journal of Reliability, Quality and Safety Engineering,* Vol. 7, No 4, pp. 301-308, 2000.

6. Paramonov, Yu., Kuznetsov, A. Inspection program development for fatigue crack growth model with two random parameters, *Aviation*, Vol.12, No 2, 2008, pp. 33-40,

7. Gertsbakh, I. *Reliability theory. With application to preventive maintenance*. Berlin, Heidelberg. New York: Springer-Verlag, 2000.