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THE CONTROLLED SEMI-MARKOV PROCESSES IN PROBLEM

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Controlled semi-Markov process $X(t)$ with a finite set of states E and an additive functional (an accumulative income) is considered. Probability measures $G_i(B)$, $i \in E$, determine a homogeneous Markov randomized control strategy. The paper's aim is to find the conditional extreme of the considered functional (the conditions on the functions $G_i(B)$, $i \in E$) and determine the structure of the distributions $G_i(B)$, $i \in E$, which prove corresponding extreme. Various models of the reliability, redundancy maintenance and queueing systems are considered as examples.

Keywords: *controlled semi-Markov process, accumulative functional, extremes*

1. Definition of the controlled semi-Markov process

The object of research is the controlled semi-Markov process $X(t)$ with a finite set of states $E=\{1,2,...,N\}$. The such semi-Markov process $X(t)$ is defined by a Markov chain (ξ_n, θ_n, u_n), $n \geq 0$, where $\xi_n \in E$, $\theta_n \in R^*=[0,\infty)$, $u_n \in U_i$, U_i are some space of control parameters which σ-algebra A_i of their subsets, $i \in E$. The Markov chain is determined by the transition characteristics (semi-Markov kernel)

$$P\{\xi_n = j, \theta_n < t, u_n \in B / \xi_{n-1} = i, \theta_{n-1} = \tau, u_{n-1} = u\} = P\{\xi_n = j, \theta_n < t, u_n \in B / \xi_{n-1} = i\} = Q_{ij}^*(t, B)$$

where $t \in R^*$, $B \in A_i$, $i,j \in E$, $n \geq 1$, and by some initial distribution $P\{\xi_0=i\}=p_i$, $i \in E$.

Then the controlled semi-Markov process $X(t)$ can be defined as the pair

$$X(t)=(\xi(t), u(t)), \quad \xi(t)=\xi_{\nu(t)-1}, \; u(t)=u_{\nu(t)},$$

where $\nu(t)= inf\,(n: \sum_{k \leq n} \theta_k > t)$, $\theta_0=0$ is the counting process. The semi-Markov kernel $Q_{ij}^*(t, B)$ defines a family of probability measures on the measurable spaces (U_i, A_i) $G_i(B) = \sum_{j \in E} Q_{ij}^*(\infty, B)$, $i \in E$, $B \in A_i$.

Since for any $j \in E$ and $t \in R^*$ $G_i(B) \geq Q_{ij}^*(t, B)$, there exist functions $Q_{ij}(t,u)$ such that

$$Q_{ij}^*(t, B) = \int_B Q_{ij}(t,u) G_i(du) \qquad (1)$$

The functions $Q_{ij}(t,u)$ can be regarded as the conditional probability

$$Q_{ij}(t,u)=P\{\xi_n=j, \; \theta_n < t \,/ \xi_{n-1}=i, \; u_n=u\}.$$

Thus, a controlled semi-Markov process can be determined by a family of matrices $(Q_{ij}(t,u))$, a set of probability measures $G_i(B)$, and an initial distribution p_i, $i,j \in E$.

2. The accumulative functional

We will denote by $\zeta(t)$ the additive accumulative income for the time interval $(0,t)$ and define $S_i(t)=\mathrm{E}(\zeta(t)/\xi(0)=i)$ as the conditional mathematical expectation of the accumulated income $\zeta(t)$ for the time interval $(0,t)$ provided that at the moment zero process $\xi(0)$ start from the state $i \in E$. The function $S_i(t)$ is determined by the functions $R_{ij}(x,u)$ which are the conditional mathematical expectations of the accumulated income for interval $(\tau, \tau+x)$ provided that at the moment of time τ the process $\xi(t)$ transits to the state i $\{\xi(\tau)=i\}$, stays in this state a time x $\{\xi(t)=i \text{ for } t\in(\tau,\tau+x)\}$, transits to the state j at the moment of time $\tau+x$. $\{\xi(\tau+x)=j\}$ and at the moment of the time τ was took controlled parameter u.

The object of study is $\lim_{t\to\infty}\frac{S_i(t)}{t}=S_i$ (if that limit exists).

3. Formulation of the problem

The following two problems arise for the controlled semi-Markov process $X(t)$. First we have to clarify how the functional S_i depends on the probability measures $G_i(B)$, $i\in E$, which determine a homogeneous Markov randomized control strategy. Secondly, we have to find the conditional extreme of these functions (the conditions on the functions $G_i(B)$, $i\in E$) and determine the structure of the distributions $G_i(B)$, $i\in E$, which prove this extreme.

4. The main results

In what follows we use the notation

$\hat{G}=(G_1,G_2,\ldots,G_N)$, $\hat{G}_i=(G_1,G_2,\ldots,G_{i-1},G_{i+1},\ldots,G_N)$,

$$p_{ij}=Q_{ij}^{*}(\infty,U_i)=\int_{U_i}Q_{ij}(\infty,u)G_i(du) \tag{2}$$

$Q_{ij}^{*}(\infty,U_i)=p_{ij}F_{ij}(t)$ for $p_{ij}>0$, $i,j\in E$,

P denotes the matrix with elements p_{ij}.

It is known [1] that if the embedded Markov chain with the transition matrix P is ergodic, if for any $j\in E$ there exists $i\in E$ such that $F_{ij}(t)$ determines a non-lattice distribution, and $\int_0^\infty t dF_{ij}(t)<\infty$, $i,j\in E$, then for any $i\in E$, as $t\to\infty$, $S_i(t)=St+o(1)$,
where

$$S=\frac{\sum_{i\in E}\sigma_i\pi_i}{\sum_{i\in E}m_i\pi_i} \tag{3}$$

$$\sigma_i=\sum_{i\in E}\int_0^\infty\int_{u\in U_i}R_{ij}(t,u)d_tQ_{ij}(t,u)G_i(du) \tag{4}$$

$$m_i=\sum_{i\in E}\int_0^\infty\int_{u\in U_i}t d_tQ_{ij}(t,u)G_i(du),\ i,j\in E \tag{5}$$

and the row vector $\bar{\pi}=(\pi_1,\pi_2,\ldots,\pi_N)$ is the stationary distribution of the states of the embedded Markov chain with the transition matrix $P=(p_{ij})=(Q_{ij}^{*}(\infty,U_i))$ which satisfies the relations

$$\bar{\pi} = \bar{\pi}P, \ \sum_{k \in E} \pi_k = 1 \tag{6}$$

***Theorema* 1.** [2,3] *If for a controlled semi-Markov process with finite set of states the embedded Markov chain is argotic, for any $j \in E$ there exists $i \in E$ such that $F_{ij}(t)$ determines a non-lattice distribution, and $m_{ij} < \infty$, $i,j \in E$, then the additive accumulative functional J is a linear-fractional functional with respect to $G_i(B)$, that is*

$$S(\bar{G}) = \frac{S_1(\bar{G})}{S_2(\bar{G})} \quad , \tag{7}$$

where

$$S_i(\bar{G}) = \int_{U^{(N)}} A_i(u_1, u_2, ..., u_N) G_1(du_1) G_2(u_2) ... G_N(du_N)$$

U(N) is a direct product of the control space U_k, $k \in E$.

***Lemma**. If the maximum of the linear-fractional functional (7) over the set V exists, then*

$$\{\bar{\Phi} : S(\bar{\Phi}) = \max_{\bar{G} \in V} S(\bar{G}) = c\} = \{\bar{\Phi} : L(\bar{\Phi}) = 0\} = H_1 \bigcup H_2,$$

where

$$H_i = \{\bar{\Phi} : L(\bar{\Phi}) = \max_{\bar{G} \in V_i} L(\bar{G}) = 0\}, \ V_i = V \bigcap W_i, i = 1, 2,$$

$$L(\bar{G}) = \int_{U^{(N)}} C(u_1, u_2, ..., u_N) G_1(du_1) G_2(u_2) ... G_N(du_N),$$

$$C(u_1, u_2, ..., u_N) = A_1(u_1, u_2, ..., u_N) - cA_2(u_1, u_2, ..., u_N).$$

$$W_1 = \{\bar{G} : S_2(\bar{G}) > 0\}, \ W_2 = \{\bar{G} : S_2(\bar{G}) < 0\},$$

Lemma gives us no possibility to find the maximum directly, since the constant *c* is unknown. However, if among the extreme distribution functions of the linear functional $H_1 \cup H_2$ there are functions of a special class, then we can search the extreme of the linear-fractional functional over this class of distributions only.

For example, the following assertion [2,3] can be easily proved using *Lemma*.

***Theorem* 3.** *If the maximum of the linear-fractional (7) exists and $A_2(u_1,u_2,...,u_N)$ is positive for $(u_1,u_2,...,u_N) \in U^{(N)}$, then*

$$\max_{\bar{G} \in V} S(\bar{G}) = \max_{u_i \in U_i, i \in E} \frac{A_1(u_1, u_2, ..., u_N)}{A_2(u_1, u_2, ..., u_N)}, \tag{8}$$

that is, the optimal distribution functions G_i, $i \in E$ can be found in class D of degenerate distribution functions, if $D \subseteq V$.

5. The examples

By this model we study the controlled semi-Markov queues and conditions maintenance of duplicated systems.

5.1. Model of reliability

We consider a complex system of *N* components (basis element and *N-1* back-up components, which do not come out of action, if they are found at the places of back-up components). Let ξ_k be the

lifetime *k-th* component, when he is found at the place of basis element, $F_k(t)=P(\xi_k<t)$, $k=1,2,...,N$. The system comes out of action at moment, when a non serviceable element is found at the place of basis component, though there are back-up components. For increase of lifetime of system we replace working component on a back-up component. Denote η_k the time before replacement of *k-th* element, $G_k(t)=P(\eta_k<t)$. Then we have $min\ (\xi_k,\eta_k)$ the time of work of *k-th* element. If $\eta_k<\xi_k$ the next element begin to work, if $\eta_k>\xi_k$ our system fails. Thus we can write the time life of system with elements $s,s+1,...,N$

$$\zeta_s=min(\xi_s,\eta_s)+\sum_{k\in\{s,s+1,...,N-1\}} min(\xi_{k+1},\eta_{k+1})I\{\eta_1<\xi_1\}I\{\eta_2<\xi_2\}...I\{\eta_k<\xi_k\},$$

where $I\{\eta_k<\xi_k\}$ denotes the indicator function of event $\{\eta_k<\xi_k\}$.

We want to definite the distributions $G_k(t)$, $k=1,2,...,N$, which prove the maxima of functional $E\zeta_1=J_1(G_1,G_2,...,G_N)$.

The optimal distributions $G^*_k(t)=0$, if $t\le t_k$, and $G^8_k(t)=1$, if $t>t_k$. The parameters t_k, is defined recurrently: $t_N=\infty$ and t_k as the solution of equation $\lambda_k(t)=[J_{k+1}(G^*_{k+1},G^*_{k+2},...,G^*_N)]^{-1}$, if $1\le k\le N-1$, where $\lambda_k(t)$ is hazard or failure rate of the distribution $F_k(t)$. In particular, for $F_k(t)=1-exp(-\lambda_k t)$, $t\ge 0$ we have $t_k=0$ for $k<m$, $t_m=\infty$, where $m=\{k:\ \lambda_k=max_{1\le n\le N}\ \lambda_n\}$.

5.2. Model of redundancy maintenance [2]

We study a system of two equivalent components (a basis element and a back-up element, which do not come out of action, if his is found at the place of back-up element).

Let ξ be the lifetime of element, when his is found at the place of basis element, $F(t)=P(\xi<t)$, and suppose that exists the time ζ before the basis element breakdown is shown itself, $\Psi(t)=P(\zeta<t)$. The random variables ξ and ζ are independent random variables.

The damaged component is repaired one worker, we shall call this repair as repair after failure. In model it is possible two kinds of preventive repair: scheduled repair and non scheduled repair. We denote γ_k the time of repair of *k-th* kind, $\Phi_k(t)=P(\gamma_k<t)$, $k=1,2,3$. The all kinds of repair make element new. Now write process of maintenance. Let be at moment $t=0$ one element begin to work and for other element is begun *k-th* kind repair. We fix the random moment η_k of scheduled repair, $G_k(t)=P(\eta_k<t)$. The next repair is begun at moment $ma\ x(\gamma_k,\ min\ (\eta_k,\xi+\zeta))$. The next repair will be scheduled preventive repair if $\gamma_k\le\eta_k\le\xi$, the next repair will be non scheduled preventive repair if $\eta_k\le\gamma_k\le\xi$, and the next repair will be of repair after failure if $max\ (\gamma_k,\eta_k)>\xi$.

Denote $0<t_1<t_2<...<t_n<...$ a sequence of moments of time when one element begin to work and for other element is begun repair. We construct semi-Markov process $\xi(t)$ assuming $\xi(t)=k$ for $t_n\le t<t_{n+1}$ if at moment t_n is begun *k-th* kind repair. The optimal control distribution functions can be found in class of degenerate distribution functions, $P(\eta_k=u_k)=1$, $k=1,2,3$ and algorithm of effectiveness or reliability calculation is defined by formulas (10) and (3)-(6) (for example, complete availability, economical (cost) effectiveness and other additive characteristics)

5.3. Queuing system $M|G_k|1\,|\,N$ [5]

We control by the service time (the dependence on the state of the system - number of finding at system demands).

The controlled semi-Markov process $X(t)=(\xi(t),u(t))$ is defined by equalities

$$\xi(t)=\xi(t_k),\ u(t)=\eta_n,\ \text{if}\ t_k\le t<t_{k+1},$$

where we denote $0<t_1<t_2<...<t_n<...$ a sequence of moments when demands leave queuing system, $\xi(t_k)$ - a number of demands at moment t_k, η_n- a service time if at system is found n demands ($\xi(t_k)=n$), $G_n(x)=P(\eta_n<x\,/\,\xi(t_k)=n)$- a conditional distributions of the service times, $n=0,1,...,N$..

We study the additive accumulative income. We receive positive income after service of demand and our income is negative if the server work, if the demand wait of beginning of service, if arriving demand can not go in system or if system is empty.

The optimal distribution functions of the service time can be degenerate, $P(\eta_n=\tau_n\,/\,\xi(t_k)=n)=1$, and parameters τ_n, $n=1,2,...,N$ are defined by (10) and (3)-(6).

5.4. Queuing system $GI_k\,|\,M\,|\,n\,|\,N$

Usually suppose that recurrent flow arrive at a system which n servers and N waiting places, the distribution function $G(t)$ of interval between the moments of the arrival do not depend on state of system.

We shell control by the interval between the moments of the arrival and select this interval (the distribution function) depending on number of demands at queuing system.

Let $0<t_1<t_2<...<t_n<...$ be moments of the arrival. The first component $\xi(t)$ of controlled semi-Markov process is defined by equality $\xi(t)=\xi(t_n)$ if $t_n \le t<t_{n+1}$ and $\xi(t_n)$- number of demands at moment t_n. The second component $u(t)$ is defined by equality $u(t)=\eta$ if $t_n \le t<t_{n+1}$, where η is moment of arrival of following demand which conditional distribution $G_k(t)=P(\eta<t \mid \xi(t_n)=k)$.

We research problem of optimal select of distribution functions $G_n(t)$, $n=1,2,...,n+N$. The additive accumulative functional is defined analogously 5.3. The optimal distribution functions of arrival moments can be degenerate $P(\eta=\tau_n \mid \xi(t_k)=n)=1$, $n+1,2,...,n+N$, and parameters τ_n is defined as the maximum's point of function (10). The algorithm of calculation of functions $A_k(u_1,u_2,...,u_{n+N})$, $k=1,2$ is defined by equalities (3)-(6).

5.5. Queuing system $GI \mid M \mid n^* \mid 0$

For this queuing system we control by number of working servers. Parameter n^* mean maximum of number of working servers. Suppose moment t_k-0 is a arrival moment. If at this moment there is free working server arrival demand go in system. If at this moment there is not free working server the arrival demand is lost. If at queuing system are found m demands at Markov's moments $0<t_1<t_2<...<t_n<...$we select or to keep m working servers (control d_0) or to keep $m+1$ working servers (control d_1). Remark that on interval $(t_k,t_{k+1}$-$0)$ can be go only one demand. Denote $P(d=d_j \mid \xi(t_k)=m)=p_{mj}$, $j=1,2$, $m=1,2,...,n$-1, $p_{m0}+p_{m1}=1$ and $p_{n0}=1$. We want to define optimal parameters p_{nj}. Clearly that optimal parameter p_{m0} is equal or 0 or 1. The algorithm of calculation of optimal parameters p_{m0} is defined by (10) and (3)-(6).

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