STABILITY CONDITIONS FOR MARKED POINT PROCESSES WITH QUASITOEPLITZ SELECTED DISCRETE COMPONENT

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It is important for practical needs to discover conditions for existence of stationary distributions of queue length, waiting time and computation of different service characteristics for systems with unreliable server. Real-world flows of customers have varying stochastic structure which can be seen as a presence of some random environment. Marked point processes with selected discrete components arise as a natural mathematical model of unreliable service process in random environment. These selected discrete components have some general properties which help investigating stationarity conditions. Example application to a single-line system with unreliable server and non-ordinary input flow is considered.

Keywords: *unreliable server, queuing system in random environment, stationary distribution, quasitoeplitz chains, iterative-dominating approach*

1. Service with Unreliable Server in Random Environment

Consider a single-line queuing system with unreliable server. Let input flow be modulated by a random environment with $W<\infty$ states $e^{(1)}$, $e^{(2)}$, …, $e^{(W)}$. At state $e^{(k)}$, $k=1$, 2, …, W, customers arrive in groups of random size and wait in infinite capacity buffer. The groups form a Poisson flow with intensity $\lambda^{(k)}$. A group contains c customers with probability $p_c^{(k)}$, $c=1$, 2, …. We assume that the generating function $f^{(k)}(z)=\sum_{c=1}^{\infty}p_c^{(k)}z^c$ converges in a circle $|z|<1+\varepsilon$ for an $\varepsilon>0$. A server failure occurs during service with probability p, $0<p<1$, then the customer after interrupted service returns back to the buffer for a new complete service later. A server failure doesn't occur with probability $1-p$ and the customer leaves the queuing system after the service. Denote by $B_0(t)$ the distribution function of a service duration given that no server failure occurred, $B_0(+0)=0$, and by $B_1(t)$ the distribution function of a service and repair duration given that a server failure occurred. Environment state changes only at epochs of the server readiness for service. Homogenous irreducible Markov chain is chosen as the mathematical model for the environment. The chain can be either periodic or aperiodic. Let $a_{l,k}$ be the probability of switching from a state $e^{(l)}$ into a state $e^{(k)}$, l, $k\in\{1,2,\dots,W\}$.

Assume a probability space $(\Omega,\mathbf{F},\mathrm{P})$ is given, where Ω is a set of elementary outcomes, $\mathbf{F}$ is a σ-algebra of events $A\subset\Omega$, P is a probability measure on $\mathbf{F}$, E denotes the mathematical expectation with respect to P. All random objects are defined or constructed on this probability space.

Taking into consideration discrete character of the queuing system's functioning, consider all processes during time intervals or at time instants which are generated by a random point process $\{\tau_i; i=0,1,\dots\}$ on the time axis. Put $\tau_0=0$. Let τ_i denote the $(i+1)$ th epoch of server readiness, $i=1$, 2, …. A time interval $(\tau_i,\tau_{i+1}]$ will be called the $(i+1)$ th working tact. Put $\kappa_0=0$. Denote by κ_i the number of customers in the queuing system at time instant τ_i. Let $\nu_i\in\{e^{(1)},e^{(2)},\dots,e^{(W)}\}$ denote the environment state during a time interval $(\tau_i,\tau_{i+1}]$. We shall say that the server was at the state $\Gamma^{(0)}$ during a working tact if no server failures occurred, otherwise the server was at the state $\Gamma^{(1)}$. Denote by $\Gamma_i\in\{\Gamma^{(0)},\Gamma^{(1)}\}$ the server state during the time interval $(\tau_i,\tau_{i+1}]$. Random variables Γ_0, Γ_1, … are independent identically distributed, $\mathrm{P}(\Gamma_0=\Gamma^{(0)})=1-p$, $\mathrm{P}(\Gamma_0=\Gamma^{(1)})=p$. From the mathematical point of view a servicing process with unreliable server is a marked point process $\{(\tau_i,\kappa_i,\nu_i); i=0,1,\dots\}$ with selected discrete component $\{(\kappa_i,\nu_i); i=0,1,\dots\}$.

Denote by η_i the total number of customers arrived during $(\tau_i, \tau_{i+1}]$. Define the mapping $u:\{e^{(1)}, e^{(2)}, \ldots, e^{(W)}\}\times(0,1)\to\{e^{(1)}, e^{(2)}, \ldots, e^{(W)}\}$ as

$$u(e^{(k)}, r) = \begin{cases} e^{(1)} & \text{for } 0 < r \le a_{k,1}, \\ e^{(l)} & \text{for } l = 2, 3, \ldots, W-1 \text{ and} \\ & a_{k,1} + a_{k,2} + \ldots + a_{k,l-1} < r \le a_{k,1} + a_{k,2} + \ldots + a_{k,l}, \\ e^{(W)} & \text{for } a_{k,1} + a_{k,2} + \ldots + a_{k,W-1} < r < 1. \end{cases}$$

Let $\{r_i; i = 0, 1, \ldots\}$ be independent identically distributed random variables with uniform distribution on $(0,1)$. Then the selected discrete component $\{(\kappa_i, \nu_i); i = 0, 1, \ldots\}$ satisfies the recurrent equation

$$(\kappa_{i+1}, \nu_{i+1}) = (\kappa_i + \eta_i - 1, u(\nu_i, r_i)) \tag{1}$$

To describe input flows in the nonlocal way [2] introduce the following random variables: the number $\eta_i^{(1)}$ of customers arrived after τ_i and before service begin, $\eta_i^{(1)} \in \{0, 1, \ldots\}$; the number $\eta_i^{(2)}$ of customers arrived when the server was active during $(\tau_i, \tau_{i+1}]$, $\eta_i^{(2)} \in \{0, 1, \ldots\}$; the number $\eta_i^{(3)}$ of customers redirected for the repeated service due to the server failure during $(\tau_i, \tau_{i+1}]$, $\eta_i^{(3)} \in \{0, 1\}$; the duration Δ_i of the $(i+1)$-th service (and repair) act, here $0 < \Delta_i \le \tau_{i+1} - \tau_i$. Clearly, $\eta_i = \, = \eta_i^{(1)} + \eta_i^{(2)} + \eta_i^{(3)}$. Now we introduce some properties of conditional distributions for the discrete component $\{(\eta_i, \chi_i); i = 0, 1, \ldots\}$ of a marked point process $\{(\tau_i, \eta_i, \chi_i); i = 0, 1, \ldots\}$ with customers' mark $\chi_i = (\kappa_i, \nu_i, \Gamma_i) \in \{0, 1, \ldots\} \times \{e^{(1)}, e^{(2)}, \ldots, e^{(W)}\} \times$ $\times \{\Gamma^{(0)}, \Gamma^{(1)}\}$ for the time interval $(\tau_i, \tau_{i+1}]$. These properties give the nonlocal description of the input flow. For a given value of the mark component (κ_i, ν_i) the variables $\eta_i^{(1)}$, $\eta_i^{(2)}$, $\eta_i^{(3)}$ are conditionally independent and have a simple form of conditional distributions. Probability $\mathrm{P}(\eta_i^{(1)} = c \mid \kappa_i = x, \nu_i = e^{(k)}, \Gamma_i = \Gamma^{(j)})$ equals 1 for $j = 0$, 1, $c = 0$ and $x = 1$, 2, …, and equals $p_c^{(k)}$ for $j = 0$, 1, $x = 0$ and $c = 1$, 2, …. Here $\{\kappa_i = 0, \eta_i^{(1)} = 0\}$ is the impossible event. For $t \ge 0$, $x = 0$, 1, … and $k = 1$, 2, …, W define functions $P(t; x, k)$ from the series expansion

$$\exp\{\lambda^{(k)} t (f^{(k)}(z) - 1)\} = \sum_{x=0}^{\infty} z^x P(t; x, k)$$

and put

$$\varphi(x; k, j) = \int_0^{\infty} P(t; x, k)\, dB_j(t), \quad j = 0, 1.$$

The formula for $P(t; x, k)$ is simple in some situations, for example when $p_1^{(k)} + p_2^{(k)} = 1$ (cf. [3]). Now $\mathrm{P}(\eta_i^{(2)} = x \mid \kappa_i = 0, \nu_i = e^{(k)}, \Gamma_i = \Gamma^{(j)}) = \varphi(x; k, j)$ for $j = 0$, 1, $x = 0$, 1, …. and $k = 1$, 2, …, W. Finally, $\mathrm{P}(\eta_i^{(3)} = j \mid \kappa_i = 0, \nu_i = e^{(k)}, \Gamma_i = \Gamma^{(j)}) = 1$. Denote by $\sigma(\cdot)$ the σ-algebra generated by the set $\{\cdot\}$ of elements listed in curly brackets. Suppose the equation

$$\begin{aligned} &\mathrm{P}(\eta_i^{(1)} = c, \eta_i^{(2)} = x, \eta_i^{(3)} = w \mid \sigma(\{(\kappa_{i_1}, \nu_{i_1}, \Gamma_{i_1}); i_i = 0, 1, \ldots, i\})) = \\ &\quad = \mathrm{P}(\eta_i^{(1)} = c, \eta_i^{(2)} = x, \eta_i^{(3)} = w \mid \sigma(\kappa_i, \nu_i, \Gamma_i)) \end{aligned} \tag{2}$$

holds for all integer nonnegative numbers c, x and for every $w \in \{0, 1\}$. Define $A_i(x, k) =$ $= \{\omega : \kappa_i = x, \nu_i = e^{(k)}\}$. Using equations (1), (2) it is easy to prove the following lemma.

Lemma 1. *Two-dimensional random sequence* $\{(\kappa_i, \nu_i); i = 0, 1, \ldots\}$ *is a Markov chain given the initial vector* (κ_0, ν_0) *with transition probability* $\mathrm{P}(A_{i+1}(y, k) \mid A_i(x, l))$ *equal to:*

$a_{l,k}(1-p)p_1^{(l)}\varphi(0;l,0)$ *for* $x=0$ *and* $y=0$*;*

$a_{l,k}\left(p\sum_{c=1}^{y}p_c^{(l)}\varphi(y-c;l,1)+(1-p)\sum_{c=1}^{y+1}p_c^{(l)}\varphi(y-c+1;l,0)\right)$ *for* $x=0$ *and* $y\geq 1$*;*

$a_{l,k}(1-p)\varphi(0;l,0)$ *for* $x>0$ *and* $y=x-1$*;*

$a_{l,k}(p\varphi(y-x;l,1)+(1-p)\varphi(y-x+1;l,0))$ *for* $x>0$ *and* $y\geq x$*;*

0 *for* $y<x-1$. *If the Markov chain* $\{\nu_i; i=0,1,\ldots\}$ *is periodic, then the sequence* $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$ *is periodic as well.*

Observe that for $x\geq 1$ transition probabilities depend only on the difference $y-x$ and depend on k, l as well. In this case the discrete component $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$ is called *quasitoeplitz*. This definition is in accordance with notion of toeplitz matrix [4] and quasitoeplitz transition matrix [5], [6].

2. Stationary Distribution Existence Conditions for Quasitoeplitz Selected Discrete Component

In the previous section explicit formulas were obtained for the transition probabilities of the selected component $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$. These formulas have a particular form, but they have some general features at the same time. Thus it is necessary to investigate two-dimensional irreducible Markov chain $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$ with the states set $\{0,1,\ldots\}\times\{e^{(1)},e^{(2)},\ldots,e^{(W)}\}$ and transition probabilities (we try to keep the same notation with the previous section as much as possible)

$$\mathrm{P}(A_{i+1}(y,k)\mid A_i(x,l))=\begin{cases}\widetilde{Q}_y^{(l,k)} & \text{for } x=0 \text{ and } y\geq 0,\\ Q_{y-x+1}^{(l,k)} & \text{for } x\geq 1 \text{ and } y\geq x-1,\\ 0 & \text{for } y<x-1.\end{cases} \tag{3}$$

Here $\widetilde{Q}_y^{(l,k)}\geq 0$, $Q_y^{(l,k)}\geq 0$, $\sum_{k=1}^{W}\sum_{x=0}^{\infty}\widetilde{Q}_x^{(l,k)}=1$, $\sum_{k=1}^{W}\sum_{x=0}^{\infty}Q_x^{(l,k)}=1$. For example, for the queuing system from the previous section

$\widetilde{Q}_0^{(l,k)}=a_{l,k}(1-p)p_1^{(l)}\varphi(0;l,0)$,

$\widetilde{Q}_y^{(l,k)}=a_{l,k}\left(p\sum_{c=1}^{y}p_c^{(l)}\varphi(y-c;l,1)+(1-p)\sum_{c=1}^{y+1}p_c^{(l)}\varphi(y-c+1;l,0)\right)$ for $y\geq 1$,

$Q_0^{(l,k)}=a_{l,k}(1-p)\varphi(0;l,0)$,

$Q_y^{(l,k)}=a_{l,k}(p\varphi(y-1;l,1)+(1-p)\varphi(y;l,0))$ for $y\geq 1$.

Another example of 2-dimensional quasitoeplitz Markov chain is given in [6]. Besides, in our case unlike [6] the Markov sequence $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$ can be periodic.

Lemma 2. *Suppose* $\sum_{y=0}^{\infty}\widetilde{Q}_y^{(l,k)}=\sum_{y=0}^{\infty}Q_y^{(l,k)}$ *for* k, $l=1, 2, \ldots, W$*; then the stochastic sequence* $\{\nu_i; i=0,1,\ldots\}$ *is an irreducible Markov chain given the distribution of* (κ_0,ν_0).

Proof. By the law of total probability, for any $i=0, 1, \ldots$ and k, $l=1, 2, \ldots, W$

$$\mathrm{P}(\nu_{i+1}=e^{(k)},\nu_i=e^{(l)})=\sum_{y=0}^{\infty}\mathrm{P}(A_i(0,l))\mathrm{P}(A_{i+1}(y,k)\mid A_i(0,l))+$$
$$+\sum_{x=1}^{\infty}\sum_{y=0}^{\infty}\mathrm{P}(A_i(x,l))\mathrm{P}(A_{i+1}(y,k)\mid A_i(x,l)).$$

The right-hand side equals

$$\mathrm{P}(A_i(0,l))\sum_{y=0}^{\infty}\widetilde{Q}_y^{(l,k)}+\sum_{x=1}^{\infty}\mathrm{P}(A_i(x,l))\sum_{y=x-1}^{\infty}Q_{y-x+1}^{(l,k)}.$$

Using the assumption of the theorem after obvious change of variable in the second sum finally obtain

$$\mathrm{P}(\nu_{i+1}=e^{(k)},\nu_i=e^{(l)})=\left(\sum_{x=0}^{\infty}\mathrm{P}(A_i(x,l))\right)\left(\sum_{y=0}^{\infty}Q_y^{(l,k)}\right)=\mathrm{P}(\nu_i=e^{(l)})\left(\sum_{y=0}^{\infty}Q_y^{(l,k)}\right),$$

$$\mathrm{P}(\nu_{i+1}=e^{(k)}\mid\nu_i=e^{(l)})=\sum_{y=0}^{\infty}Q_y^{(l,k)}.$$

For any l_0, l_1, …, $l_{i-1}=1,\ 2,\ \ldots,\ W$ in a similar manner

$$\mathrm{P}(\nu_{i+1}=e^{(k)}\mid\nu_0=e^{(l_0)},\nu_1=e^{(l_1)},\ldots,\nu_{i-1}=e^{(l_{i-1})},\nu_i=e^{(l)})=\sum_{y=0}^{\infty}Q_y^{(l,k)}.$$

To conclude the proof we shall demonstrate that any two states $e^{(k)}$, $e^{(l)}$ communicate. Since for any integer nonnegative x, y the states $(x,e^{(k)})$ and $(y,e^{(l)})$ communicate, there exists a sequence

$$(x,e^{(k)})\to(x_1,e^{(k_1)})\to(x_2,e^{(k_2)})\to\ldots\to(x_g,e^{(k_g)})\to(y,e^{(l)})$$

such that

$$\mathrm{P}(\nu_{i+g+1}=e^{(l)}\mid\nu_i=e^{(k)})=\left(\sum_{x=0}^{\infty}Q_x^{(k,k_1)}\right)\left(\sum_{x=0}^{\infty}Q_x^{(k_1,k_2)}\right)\cdots\left(\sum_{x=0}^{\infty}Q_x^{(k_g,l)}\right)>0.$$

From Equation (3) recurrent with respect to i relations can be obtained

$$\mathrm{P}(A_{i+1}(y,k))=\sum_{l=1}^{W}\left(\mathrm{P}(A_i(0,l))\widetilde{Q}_y^{(l,k)}+\sum_{x=1}^{y+1}\mathrm{P}(A_i(x,l))Q_{y-x+1}^{(l,k)}\right).\qquad(4)$$

Define

$$R^{(l,k)}(z)=\sum_{x=0}^{\infty}z^xQ_x^{(l,k)},$$

$$\widetilde{R}^{(l,k)}(z)=\sum_{x=0}^{\infty}z^x\widetilde{Q}_x^{(l,k)},$$

$$\Psi_i(z,k)=\sum_{x=0}^{\infty}z^x\,\mathrm{P}(A_i(x,k)).$$

The series converge for at least $|z|\le 1$. From (4) we obtain

$$\Psi_{i+1}(z,k)=\sum_{l=1}^{W}\left(z^{-1}\Psi_i(z,l)R^{(l,k)}(z)+\Psi_i(0,l)(\widetilde{R}^{(l,k)}(z)-z^{-1}R^{(l,k)}(z))\right).\qquad(5)$$

In what follows we assume that the series $R^{(l,k)}(z)$ and $\widetilde{R}^{(l,k)}(z)$ converge in a circle $|z|\le 1+\varepsilon_1$ for some $\varepsilon_1>0$. It follows from (5) that generating functions $\Psi_{i+1}(z,k)$, $k=1,\ 2,\ \ldots,\ W$, converge in the circle, as soon as $\Psi_0(z,k)$, $k=1,\ 2,\ \ldots,\ W$, converge in this circle. It follows from the Cauchy formula that $\mathrm{E}\,\kappa_i=\frac{d}{dz}(\Psi_i(z,1)+\Psi_i(z,2)+\ldots+\Psi_i(z,W))\big|_{z=1}<\infty$ for $i=0,\ 1,\ \ldots$.

For the irreducible denumerable Markov chain $\{(\kappa_i,\nu_i);i=0,1,\ldots\}$ two alternatives exist [7]. The chain either possesses a unique stationary probability distribution

$$\{\pi(x,k):k=1,2,\ldots,W;x=0,1,\ldots\},\qquad(6)$$

or

$$\lim_{i\to\infty} \mathrm{P}(A_i(x,k)) = 0 \tag{7}$$

for every $k=1,\ 2,\ \ldots\ W$, $x=0,\ 1,\ \ldots$ and independently of the distribution of (κ_0,ν_0).

Theorem 1. *If the stationary distribution doesn't exist, then the sequence* $\{\mathrm{E}\,\kappa_i; i=0,1,\ldots\}$ *is unbounded.*

In other words, it is sufficient that the sequence $\{\mathrm{E}\,\kappa_i; i=0,1,\ldots\}$ is bounded for the stationary distribution of the chain $\{(\kappa_i,\nu_i); i=0,1,\ldots\}$ to exist.

Proof of Theorem 1. For a fixed i and any natural N the following estimates hold:

$$\mathrm{E}\,\kappa_i = \sum_{k=1}^{W}\sum_{x=0}^{\infty} x\,\mathrm{P}(A_i(x,k)) \ge \sum_{k=1}^{W}\sum_{x=N+1}^{\infty} x\,\mathrm{P}(A_i(x,k)) \ge$$

$$\ge (N+1)\sum_{k=1}^{W}\sum_{x=0}^{\infty}\mathrm{P}(A_i(x,k)) = (N+1)\left(1-\sum_{k=1}^{W}\sum_{x=0}^{N}\mathrm{P}(A_i(x,k))\right).$$

Using (7) we have $\mathrm{E}\,\kappa_i \ge N$ for $i \ge i_N$.

Denote by I_W the $W\times W$-identity matrix. In accordance with the previous section put $a_{l,k}=\sum_{x=0}^{\infty} Q_x^{(l,k)}$, then $\sum_{k=1}^{W} a_{l,k}=1$ and the matrix $A=(a_{l,k})_{l,k=1,2,\ldots,W}$ is stochastic. By theorem 2.1 [8], finite limit exists for the sequence $n^{-1}(I_W + A + A^2 + \ldots + A^{n-1})$, $n=1,\ 2,\ \ldots$, and this limit is also a stochastic matrix. Let $\delta_{l,k}$ be the Kronecker delta, define $A^n = (a_{l,k}^{(n)})_{l,k=1,2,\ldots,W}$,

$$\rho_l = \sum_{k=1}^{W}\left.\frac{dR^{(l,k)}(z)}{dz}\right|_{z=1} \ge 0,$$

$$\widetilde{\rho}_l = \sum_{k=1}^{W}\left.\frac{d\widetilde{R}^{(l,k)}(z)}{dz}\right|_{z=1} \ge 0,$$

$$\alpha_{l,k} = \lim_{n\to\infty} n^{-1}(\delta_{l,k} + a_{l,k} + a_{l,k}^{(2)} + \ldots + a_{l,k}^{(n-1)}).$$

Theorem 2. *For the stationary distribution (6) existence it is sufficient that*

$$\max_{1\le l\le W}\{\alpha_{l,1}\rho_1 + \alpha_{l,2}\rho_2 + \ldots + \alpha_{l,W}\rho_W\} < 1 \tag{8}$$

Proof. First we demonstrate that the sequence $\{\mathrm{E}\,\kappa_i; i=0,1,\ldots\}$ is bounded for some probability distribution of (κ_0,ν_0) and then address Theorem 1. For a positive integer q applying equation (5) q times we get

$$\sum_{k=1}^{W}\Psi_{i+q}(z,k) = \sum_{l_q=1}^{W} z^{-q}\Psi_i(z,l_q)\sum_{l_{q-1}=1}^{W}\cdots\sum_{l_1=1}^{W}\sum_{k=1}^{W} R^{(l_q,l_{q-1})}(z)\cdots R^{(l_1,k)}(z) +$$

$$+\sum_{l_1=1}^{W}\Psi_{i+q-1}(0,l_1)\left(\sum_{k=1}^{W}\widetilde{R}^{(l_1,k)}(z) - z^{-1}\sum_{k=1}^{W}R^{(l_1,k)}(z)\right) +$$

$$+\sum_{l_2=1}^{W} z^{-1}\Psi_{i+q-2}(0,l_2)\left(\sum_{l_1=1}^{W}\sum_{k=1}^{W}\widetilde{R}^{(l_2,l_1)}(z)R^{(l_1,k)}(z) - z^{-1}\times\right.$$

$$\left.\times\sum_{l_1=1}^{W}\sum_{k=1}^{W}R^{(l_2,l_1)}(z)R^{(l_1,k)}(z)\right) + \ldots + \sum_{l_q=1}^{W} z^{-q+1}\Psi_i(0,l_q)\times$$

$$\times\left(\sum_{l_{q-1}=1}^{W}\cdots\sum_{l_1=1}^{W}\sum_{k=1}^{W}\sum_{k=1}^{W}\widetilde{R}^{(l_q,l_{q-1})}(z)\cdots R^{(l_2,l_1)}(z)R^{(l_1,k)}(z) -\right.$$

$$\left.- z^{-1}\sum_{l_{q-1}=1}^{W}\cdots\sum_{l_1=1}^{W}\sum_{k=1}^{W}\sum_{k=1}^{W}R^{(l_q,l_{q-1})}(z)\cdots R^{(l_2,l_1)}(z)R^{(l_1,k)}(z)\right).$$

Find the derivative of the coefficient of $\Psi_i(z,l_q)$ at $z=1$. After some manipulations we have:

$$\frac{d}{dz}\left(z^{-q} \sum_{l_{q-1}=1}^{W} \cdots \sum_{l_1=1}^{W}\sum_{k=1}^{W} R^{(l_q,l_{q-1})}(z)\cdots R^{(l_1,k)}(z) \right)\Bigg|_{z=1} =$$

$$= q\left(\sum_{k=1}^{W} \rho_k q^{-1} (\delta_{l_q,k} + a_{l_q,k} + a^{(2)}_{l_q,k} + \ldots + a^{q-1}_{l_q,k}) - 1 \right).$$

Since for $l = 1,\ 2,\ \ldots,\ W$

$$\lim_{q\to\infty} \sum_{k=1}^{W} \rho_k q^{-1}(\delta_{l,k} + a_{l,k} + a^{(2)}_{l,k} + \ldots + a^{q-1}_{l,k}) = \sum_{k=1}^{W} \rho_k \alpha_{l,k} < 1,$$

a positive integer q_0 exists such that

$$q_0\left(\sum_{k=1}^{W} \rho_k q_0^{-1} (\delta_{l_{q0},k} + a_{l_{q0},k} + a^{(2)}_{l_{q0},k} + \ldots + a^{q_0-1}_{l_{q0},k}) - 1 \right) < 0 .$$

Put for some $z_0 \in (1, 1+\varepsilon_1)$

$$R_+ = z_0^{-1} \max_{1\le l\le W} \left\{ \sum_{l_{q0-1}=1}^{W} \cdots \sum_{l_1=1}^{W}\sum_{k=1}^{W} R^{(l,l_{q0-1})}(z_0)\cdots R^{(l_1,k)}(z_0) \right\},$$

then $R_+ < 1$. Assume the random vector (κ_0, ν_0) takes values from the set $\{(0,e^{(1)}),\ (0,e^{(2)}), \ldots, (0,e^{(W)})\}$ almost surely. Then $\Psi_0(z,k)$ is a constant, $\Psi_0(z,k) < \infty$. A nonnegative number B can be taken such that

$$B > \sum_{l_1=1}^{W} \Psi_{i+q_0-1}(0,l_1) \left(\sum_{k=1}^{W} \widetilde{R}^{(l_1,k)}(z) - z^{-1} \sum_{k=1}^{W} R^{(l_1,k)}(z) \right) +$$

$$+ \sum_{l_2=1}^{W} z^{-1} \Psi_{i+q_0-2}(0,l_2) \left(\sum_{l_1=1}^{W}\sum_{k=1}^{W} \widetilde{R}^{(l_2,l_1)}(z) R^{(l_1,k)}(z) - z^{-1} \times \right.$$

$$\left. \times \sum_{l_1=1}^{W}\sum_{k=1}^{W} R^{(l_2,l_1)}(z) R^{(l_1,k)}(z) \right) + \ldots + \sum_{l_q=1}^{W} z^{-q_0+1} \Psi_i(0,l_{q_0}) \times$$

$$\times \left(\sum_{l_{q0-1}=1}^{W} \cdots \sum_{l_1=1}^{W}\sum_{k=1}^{W}\sum_{k=1}^{W} \widetilde{R}^{(l_{q0},l_{q0-1})}(z) \cdots R^{(l_2,l_1)}(z) R^{(l_1,k)}(z) - \right.$$

$$\left. - z^{-1} \sum_{l_{q0-1}=1}^{W} \cdots \sum_{l_1=1}^{W}\sum_{k=1}^{W}\sum_{k=1}^{W} R^{(l_{q0},l_{q0-1})}(z) \cdots R^{(l_2,l_1)}(z) R^{(l_1,k)}(z) \right).$$

Thus for $i = 0,\ 1,\ \ldots$ we have

$$0 \le \sum_{k=1}^{W} \Psi_{i+q_0}(z_0,k) < R_+ \sum_{k=1}^{W} \Psi_i(z_0,k) + B < \infty .$$

By definition put

$$M_0 = \sum_{k=1}^{W} \Psi_0(z_0,k),$$

$$M_1 = \sum_{k=1}^{W} \Psi_1(z_0,k),\ \ldots,$$

$$M_{q_0-1} = \sum_{k=1}^{W} \Psi_{q_0-1}(z_0,k),$$

$$M_{i+q_0} = R_+ M_i + B\,,\ i = 0\,,\ 1\,,\ \ldots$$

Then the sequence $\{M_i; i = 0, 1, \ldots\}$ contains q_0 convergent subsequences, so it is bounded. Then sequences $\{\Psi_i(z_0, k); i = 0, 1, \ldots\}$, $k = 1,\ 2,\ \ldots,\ W$, are bounded and finally the sequence $\{\mathrm{E}\,\kappa_i; i = 0, 1, \ldots\}$ is bounded due to Cauchy formula. Now use Theorem 1 to see that the stationary distribution exists.

For our queuing system with unreliable server we obtain from definition

$$R^{(l,k)}(z) = a_{l,k}(1-p)\varphi(0;l,0) + \sum_{x=1}^{\infty} z^x a_{l,k}(p\varphi(x-1;l,1) + (1-p)\varphi(x;l,0)) =$$

$$= a_{l,k}\left((1-p)\int_0^\infty \exp\{\lambda^{(l)} t(f^{(l)}(z)-1)\}\, dB_0(t) + pz\int_0^\infty \exp\{\lambda^{(l)} t(f^{(l)}(z)-1)\}\, dB_1(t)\right),$$

$$\widetilde{R}^{(l,k)}(z) = a_{l,k}(1-p)p_1^{(l)}\varphi(0;l,0) + a_{l,k}\sum_{x=1}^{\infty} z^x \left(p\sum_{c=1}^{x} p_c^{(l)}\varphi(x-c;l,1) + \right.$$

$$\left. + (1-p)\sum_{c=1}^{x+1} p_c^{(l)}\varphi(x-c+1;l,0)\right) = a_{l,k} z^{-1} f^{(l)}(z) \times$$

$$\times\left((1-p)\int_0^\infty \exp\{\lambda^{(l)} t(f^{(l)}(z)-1)\}\, dB_0(t) + pz\int_0^\infty \exp\{\lambda^{(l)} t(f^{(l)}(z)-1)\}\, dB_1(t)\right).$$

Denote by $\mu^{(k)} = \sum_{c=1}^{\infty} c p_c^{(k)}$ the expected group size for the environment state $e^{(k)}$, by $\beta_j = \int_0^\infty t\, dB_j(t)$, $j = 0\,,\ 1$, the expected service and repair durations. It is easy to verify that

$$\rho_k = \sum_{l=1}^{W} \frac{d}{dz} R^{(k,l)}(z)\bigg|_{z=1} = \lambda^{(k)}\mu^{(k)}(p\beta_1 + (1-p)\beta_0) + p\,.$$

Since the environment by assumptions is an irreducible Markov chain, the limiting values $\alpha_{l,k} = \alpha_k$ are independent of l, thus the sufficient condition (8) has the form

$$\left(\frac{p}{1-p}\beta_1 + \beta_0\right)\sum_{l=1}^{W} \alpha_l \lambda^{(l)} \mu^{(l)} < 1\,. \tag{9}$$

This condition has a simple interpretation. The first left-hand side factor is the expected total service-and-repair time dedicated to a single customer until the customer leaves the queuing system. Notice that the number of failures associated with one customer has geometric distribution with parameter p. The second left-hand side factor is the average input intensity with respect to environment.

Theorem 3. *Assume* $\widetilde{R}^{(l,k)}(1) = R^{(l,k)}(1)$, $1 + \widetilde{\rho}_k - \rho_k > 0$, k, $l = 1,\ 2,\ \ldots\ W$, *and the stationary distribution (6) exists. Then relation (8) holds.*

Proof. Assume the stationary distribution (6) exists. Let the random vector (κ_0, ν_0) have distribution (6). This guarantees that limit exists

$$\lim_{i\to\infty} \mathrm{P}(A_i(x,k)) = \pi(x,k) > 0$$

for every $k = 1,\ 2,\ \ldots\ W$, $x = 0\,,\ 1\,,\ \ldots$. Generating functions

$$\Psi(z,k) = \sum_{x=0}^{\infty} z^x \pi(x,k)\,,\ k = 1,\ 2,\ \ldots\ W$$

satisfy equations

$$\Psi(z,k) = \sum_{l=1}^{W}\left(z^{-1}\Psi(z,l)R^{(l,k)}(z) + \Psi(0,l)(\widetilde{R}^{(l,k)}(z) - z^{-1}R^{(l,k)}(z))\right), \tag{10}$$

$$\sum_{k=1}^{W}\Psi(z,k)=\sum_{l=1}^{W}\left(z^{-1}\Psi(z,l)\sum_{k=1}^{W}R^{(l,k)}(z)+\Psi(0,l)\sum_{k=1}^{W}(\widetilde{R}^{(l,k)}(z)-z^{-1}R^{(l,k)}(z))\right). \quad (11)$$

In the left neighborhood of $z=1$ we have

$$z^{-1}\sum_{k=1}^{W}R^{(l,k)}(z)=1+(\rho_l-1)(z-1)+O((z-1)^2), \quad (12)$$

$$\sum_{k=1}^{W}(\widetilde{R}^{(l,k)}(z)-z^{-1}R^{(l,k)}(z))=(\widetilde{\rho}_l+1-\rho_l)(z-1)+O((z-1)^2). \quad (13)$$

Substitution of equations (12), (13) into equation (11) gives

$$\sum_{l=1}^{W}(\Psi(z,l)(\rho_l-1)+\Psi(0,l)(\widetilde{\rho}_l+1-\rho_l))(z-1)+O((z-1)^2)=0.$$

Dividing by $(z-1)$ and letting $z\to 1$ from the left finally obtain

$$\sum_{l=1}^{W}(\Psi(z,l)(\rho_l-1)+\Psi(0,l)(\widetilde{\rho}_l+1-\rho_l))=0.$$

Substitute $z=1$ in (10).We get

$$\Psi(1,k)=\sum_{l=1}^{W}a_{l,k}\Psi(1,l). \quad (14)$$

Since $\widetilde{R}^{(l,k)}(1)=R^{(l,k)}(1)$ for all k, $l=1,\ 2,\ \ldots\ W$, the matrix A is the transition matrix of an irreducible Markov chain $\{\nu_i; i=0,1,\ldots\}$. Recall the normalization condition

$$\Psi(1,1)+\Psi(1,2)+\ldots+\Psi(1,W)=1,$$

then equations (14) have the unique solution $\Psi(1,k)=\alpha_{l,k}=\alpha_k$. Hence,

$$\sum_{l=1}^{W}\Psi(0,l)(1+\widetilde{\rho}_l-\rho_l)=1-\sum_{l=1}^{W}\alpha_l\rho_l.$$

And the left-hand side is positive, because $\Psi(0,l)=\pi(0,l)>0$. This proves the theorem.

Applying Theorem 3 to the study of our queuing system with unreliable server observe that (9) gives both the necessary and sufficient condition of the stationary distribution existence.

Conclusions

Sufficient conditions for the stationary distribution existence in quasitoeplitz Markov chains are computationally inexpensive and can be easily checked in practical applications. At the same time their form demonstrates explicitly the averaging influence of the random environment. Still further work on necessary conditions is needed.

References

1. Zorin, A. V., Fedotkin, M. A. Optimization of doubly stochastic non-ordinary flows in time-sharing systems, *Automation and Remote Control*, Vol. 6, No. 7, 2005, pp. 1115–1124.
2. Fedotkin, M. A. Optimal control for conflict flows and marked point processes with selected discrete component, I. *Lietuvos matematikos rinkinys*, Vol. 28, No. 4, 1988, pp. 783–794.
3. Gnedenko, B. V., Kovalenko, I. N. *Introduction to queuing theory.* Moscow: Nauka, 1987. 336 p.
4. Voevodin, V. V., Kuznetsov, Yu. A. *Matrices and computations.* Moscow: Mir, 1984. 318 p.
5. Dukhovny, A. M. Markov chains with quasitoeplitz transition matrix, *Journal Appl. Math. Simul.*, Vol. 2, No. 1, 1989, pp. 71–82.
6. Dudin, A. N., Klimenok, V. I. Calculation of single-line queuing system' characteristics functioning in markovian synchronous random environment, *Avtomatika i vychislitelnaja tekhnika*, No. 1, 1997, p. 74–83.
7. Shiryaev ,A. N. *Probability. 2nd ed.* Springer, 1995. 621 p.
8. Doob, J. L. *Stochastic processes.* Wiley-Interscince, 1990. 655 p.