ESTMATION OF RISK INSURANCE IN PRACTICE OF WORK OF ACTUARIALS

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The problem of estimating various probabilistic characteristics of risk in activity of actuaries of the insurance company is considered in the article. The model of collective risk is discussed more detail. As well as in models of individual risk, in models of collective risk ruin is defined by total payments S of the insurance company of kind S= $Y_1 + Y_2 + Y_3 + \ldots + Y_\nu$, and the probability of ruin of the company are defined as P (Y1 + Y2 + Y3 + … + Y_ν > U), where U - size of actives of the company. Exact analytical estimation of this probability for any distributions of random variables ν and Y_ν are far not always possible to receive. In the article it is offered and numerically realized the algorithm of an estimation of probability of ruin of the insurance company under conditions of model of collective risk at use practically any distribution of size of damage and trivial discrete (non-negative integer) distribution of the number of insurance cases.

Keywords: *statistics, insurance, estimation, collective risk model*

1. Introduction

The estimation of financial risk of the insurance company begins with a choice and construction of reasonable mathematical models. In models of collective risk which here are discussed, the basic characteristic of a portfolio of contracts of insurance is not the number of concluded contracts N, but total number ν of insurance cases for the analyzed period. It is clear that ν is a random variable, and random variables of damage owing to insurance cases Y1, Y_2, Y_3 … Y_ν believe independent and equally distributed. As well as in models of individual risk, in models of collective risk ruin is defined by total payments S of the insurance company of kind $S = Y_1 + Y_2 + Y_3 + \ldots + Y_\nu$. The probability of ruin of the company is defined as

$$P\ (Y1 + Y2 + Y3 + \ldots + Y_\nu > U), \quad (1)$$

where U – size of actives of the company.

2. Main results

Knowledge of distribution of number of insurance cases and sizes of individual damage Y_ν allows establishing many characteristics of risk in activity of the insurance company. However it is possible to make it far not always and often depends on distributions which participate in model. Let's consider, being based on [1], for example, a case when distribution of size of individual damage of the insurance company has normal distribution with known parameters μ, σ^2, those $Y \sim N(\mu, \sigma^2)$. Thus discrete not negative distribution of number of insurance cases is set by the following table:

Table 1. Distribution of number of insurance cases

N	P(N=n)
0	0.5
1	0.2
2	0.2
3	0.1

Value S0=100 which characterizes actives of the insurance company is known also. The problem of definition of probability P (S>S0) is put, i.e. probabilities when total damage in all insurance cases will exceed actives of the company, as will lead to ruin at last. The decision of this problem can be made on the basis of the formula of full probability of a kind

$$P(S > S_0) = \sum_{i=0}^{n} P(N = i) \cdot P(X^{*i} > S_0), \quad (2)$$

where
S – total damage,
S_0 – a preset value describing actives of the company,

X^{*i} – distribution of the sum of individual damage at number composed i.

Coming back to a case of normal distribution N(μ, σ^2), it is possible to find simply enough distribution r.v. $X^{*i} = X_1 + X_2 + ... + X_i$, which has distribution $N(i \cdot \mu, i \cdot \sigma^2)$.

At concrete values $\mu = 120,\ \sigma^2 = 9, S_0 = 100$ the formula (2) turns to the formula of a kind

$$P(S > 100) = \sum_{i=0}^{3} P(N = i) \cdot P(X^{*i} > 100) =$$

$$= \sum_{i=0}^{3} P(N = i) \cdot P(X^{*i} > 100) = 0.5 \cdot 0 + 0.2 \cdot P(X > 100) + 0.2 \cdot P(X^{*2} > 100) + 0.1 \cdot P(X^{*3} > 100)$$

Clearly that r.v. X has distribution $N(120,9)$, and $X^{*2} = X_1 + X_2 \sim N(240,18)$,

$X^{*3} = X_1 + X_2 + X_3 \sim N(360,27)$.

Therefore

$$P(S > 100) = 0.2 \cdot P\left(Z > \frac{100 - 120}{3}\right) + 0.2 \cdot P\left(Z > \frac{100 - 240}{\sqrt{18}}\right) + 0.1 \cdot P\left(Z > \frac{100 - 360}{\sqrt{27}}\right)$$

Generalization of the considered approach is the finding of probability of excess of total damage S of set constant S_0 in conditions of any distribution of size of damage. Thus our problem consists of the following. Demanded distribution k random variables is offered to be found in the formula (2) numerically, being based on the formula of convolution

$$F^k_{X^*}(x) = \int_0^x F^{(k-1)}_{X^*}(x - y) f_X(y) dy, \tag{3}$$

where $F^k_{X^*}(x)$ – distribution of the sum k random variables, each of which has density of distribution of damage $f_X(y)$.

For better understanding we shall consider the following example. Let distribution of damage of the insurance company, having concrete, for example, exponential distribution $f(x) = \lambda * e^{-\lambda * x}$ with known parameter λ. Discrete distribution of the number of the insurance cases, set by Table 2 is known also.

Table 2. Distribution of number of insurance cases for the exponential distribution of damage

n	**P(N=n)**
0	0.45
1	0.25
2	0.10
3	0.08
4	0.12

As well as earlier, it is necessary to define probability of that total losses S in model of collective risk will exceed some preset value C which characterizes actives of the company. For the decision of a problem under conditions of a random number of insurance cases it is possible to take advantage, as well as in case of normal distribution, representation of a kind

$$P(S > C) = \sum_{i=0}^{n} P(N = i) P(X^i_* > C) =$$

$$= 0.25 * P(X^1_* > C) + 0.1 * P(X^2_* > C) + 0.08 * P(X^3_* > C) + 0.12 * P(X^4_* > C), \tag{4}$$

where X^i_* means distribution of the sum of i composed.

Distribution X^i_* is defined by the sum i exponential random variables that gives density of Erlang distribution [3]

$$F(x) = \begin{cases} 0, x < 0 \\ 1 - \sum_{i=0}^{l-1} \frac{(\lambda * x)^i}{i!} . e^{-\lambda x}, x \geq 0 \end{cases}$$

Realizing analytical expression (4) with use of the formula for F (x), we can receive required probability of ruin of the company, for example, at C=10 probability P (S> C) =0.048.

Consecutive application recurrent formulas (3) in Matchcad environment for the above described example, is presented in following program listing.

```
probvn(c) := | f(x) ← λ·exp(−λ·x)
             | F1(x) ← 1 − exp(−λ·x)
             | F2(x) ← ∫_0^x F1(x − y)·f(y) dy
             | F3(x) ← ∫_0^x F2(x − y)·f(y) dy
             | F4(x) ← ∫_0^x F3(x − y)·f(y) dy
             | pr ← (1 − F1(c))·p1 + (1 − F2(c))·p2 + (1 − F3(c))·p3 + (1 − F4(c))·p4
                          probvn(10) = 0.048
```

Fig 1. Realization of the formula (4) in Matchcad environment

Correctness of the decision is provided with concurrence of results both in analytical, and in case of numerical realization that is presented on the schedule for diapason of the values 1 <c <20

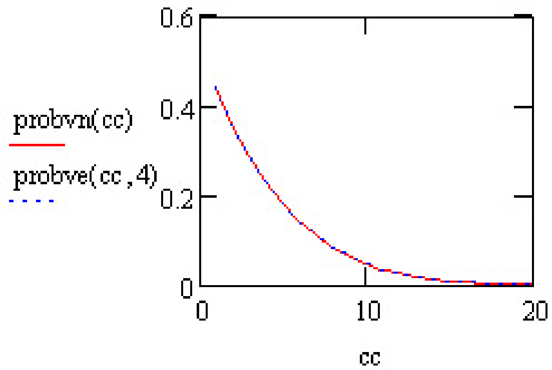

Fig 2. Concurrence of the analytical and numerical decision

The major consequence of the aforesaid is the opportunity to duplicate the considered approach with use of function of convolution for any densities which describe distribution of individual damage. At all there is no necessity to achieve analytical representation of results for the numerical approach becomes equivalent.

We shall consider one more example, where number of insurance cases has discrete distribution which is set by the following Table 3:

Table 3. Distribution of number of insurance cases

n	**p(N=n)**
0	0.6
1	0.2
2	0.2

The density of damage is described by density of distribution of a kind

$$f(x) = \begin{cases} x \cdot e^{-x}, \ x \ge 0 \\ 0, \ x < 0 \end{cases}$$

It is necessary to find P(S> 10), that is probability of that total damage S will exceed preset value S0 = 10.

To find required probability, we shall take advantage of the formula considered earlier:

$$p(S > 10) = \sum_{i=0}^{2} p(N = i) \cdot p\left(X^{*i} > 10\right),$$

where X^{*i} means distribution of the sum i composed.

So $P(S > 10) = 0.2 \cdot (1 - F(10)) + 0.2 \cdot (1 - F_2(10))$

To calculate required probability, it is necessary for us to find the function of distribution, and also the function of distribution of the sum of two random values. By integration we shall define an analytical sight of functions of distribution:

$$F(x) = \int_0^x x \cdot e^{-x} dx = -x \cdot e^{-x} - e^{-x} + 1$$

The function of distribution for the sum of two values will be defined by expression of a sight

$$F_2(x) = \int_0^x F(x - y) \cdot y \cdot e^{-y} dy = 1 - e^{-x} - x \cdot e^{-x} - \frac{x^2 e^{-x}}{2} - \frac{x^3 e^{-x}}{6}. \quad (5)$$

Having substituted in the formula (5) found functions of distributions, we can define required probability:

$$p(S > 10) = 0.2 \cdot (1 - F(10)) + 0.2 \cdot (1 - F_2(10)) = 0.00217. \quad (6)$$

The same problem is easily solved numerically with the use of recurrent formula (3). Not making comments in details technics of realization of the formula (3) in Matchcad, we shall mention, that the result will be identical to the previous result (6).

Other important and modern approach of application of the model of collective risk is the model based on formula Panjer [2]. Recommendations and some ideas on introduction of this algorithm in practice of work of insurance firm briefly will here are discussed. In particular, we shall result to more exact the formulation of the statement Panjer. Let total payments S of the insurance company are defined by expression of a sight

$$S = Y_1 + Y_2 + Y_3 + \ldots + Y_\nu, \quad (7)$$

where Y_1, Y_2, Y_3, … Y_ν - random values of losses owing to insurance cases are independent and equally distributed. Let also a random variable ν satisfy to property of recursive of a kind

$$P(\nu=n)=(a+b/n)P(\nu=n-1) \quad (8)$$

Then for model of collective risk of kind $S = Y_1 + Y_2 + Y_3 + \ldots + Y_\nu$, where ν has the specified property (8), Yi, i=1,2,… ν, positive random variables and $f_S(0) = f_\nu(0)$, it is possible to find distribution r.v. S and thus the formula takes place

$$f_S(r) = \sum_{j=1}^{r} (a + bj/r) f_Y(j) f_S(r - j), \; r = 1,2,... \quad (9)$$

where

$f_S(r) -$ density (or probability) random variable S ; $f_Y(y) -$ density (or probability) a random variable of damage Y.

Let's consider application of this formula on a concrete example, thus realization of formula (9) we shall make as program in package Matchcad.

Let's preliminary notice also, that, if a random variable ν has, for example, Poisson distribution , property of recursive (9) is easily carried out. Really, let

$$P(\nu=n) = \frac{\lambda^n e^{-\lambda}}{n!} = (0 + \lambda / n) P(\nu = n - 1), \text{ т.е. } a=0, b=\lambda. \quad (10)$$

Further we shall consider applicability and numerical realization of algorithm on a concrete example [2].

Example 1. The size of damage S has compound Poisson distribution i.e.

S = Y1 + Y2 + Y3 + … + Y_ν, thus value ν has Poisson distribution with parameter λ=12, and value Y accepts values {1, 5, 10} with corresponding probabilities {0.2, 0.3, 0.5}.

In this case the formula (9) turns to the formula of a kind

$$P(S = r) = \sum_{j=1}^{\min(r,10)} (\lambda * j / r) P_Y(j) P(S = r - j) = \frac{12}{r} \sum_{j=1}^{\min(r,10)} j P_Y(j) P(S = r - j)$$

Below the program in package Matchcad, realizing the specified formula, i.e. distribution r.v. S is presented. S is easily calculated at any r

$$PS_0 := e^{-3} \qquad fx_1 := 0.2 \qquad fx_5 := 0.3 \qquad fx_{10} := 0.5$$

$$\text{prupr}(r) := \left| \begin{array}{l} PS_0 \leftarrow e^{-12} \\ \text{for } \ j \in 1..\min(r, 10) \\ \quad PS_j \leftarrow \dfrac{12 \cdot \sum_{k=1}^{j} fx_k \cdot k \cdot PS_{j-k}}{(j)} \\ \text{sum1} \leftarrow PS_r \ \text{ if } \ r \le 10 \\ \text{for } \ j \in 11..r \qquad\qquad\qquad \text{if } \ (r > 10) \\ \quad PS_j \leftarrow \dfrac{12 \cdot \sum_{k=1}^{\min(r,10)} fx_k \cdot k \cdot PS_{j-k}}{(j)} \\ \text{sum1} \leftarrow PS_r \end{array} \right.$$

Fig 2. Realization of formula Panjer in Matchcad environment for an example 1

Conclusions

In the conclusion we shall notice, that we offer and numerically realize algorithm of an estimation of probability of ruin of the insurance company under conditions of model of collective risk at use practically any distribution of size of damage and trivial discrete distribution of number of insurance cases. The Panjer formula is lead also to numerical realization. The discussed algorithms with their numerical realization can be recommended for introduction in practice of the insurance companies work.

References

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