MODELLING OF THE MULTIPRODUCT INVENTORY PROBLEM

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The given paper considers a multiproduct inventory control model with random demands for goods. The authors propose the model with a fixed period of time between the moments of placing neighbouring orders. The order quantity is determined as the difference between the fixed stock level and the quantity of goods at the moment of ordering. It is assumed that each order may enclose only fixed kinds of products. The considered model is realized by the combination of the analytical and simulation approaches in packages MS Excel and ExtendSim correspondingly. The numerical example of the problem solving is presented. The considered model is of interest because it illustrates a real business situation.

Keywords: *inventory control, random demand, analytical method, optimisation, simulation*

1. Introduction

The inventory control problems are complex in practice.

Decision-makers need to understand that this complexity depends on their role in business and the way they choose to solve the problems. Modelling of inventory control has always been accepted by the tools, which can provide the description of those business situations, which are difficult to examine in any other way. The appropriate technique must be employed to model a different type of the inventory control system: deterministic or stochastic, single- or multi-period, single- or multi-product, and so on. There must be different ways in which the company can place and receive an order, for example, using direct delivery or using the chain "producer – wholesaler – customer".

The economic, social and technical characteristics of the problem determine the structure of the constructed model. We have to investigate the stochastic models for different situations characterized by the inventory control systems and there is a set of stochastic models available for solving the inventory control problem [1, 2].

In the given paper a multiproduct inventory control model for n products with random demand and known distributions is considered. We investigate the ordering strategy when each order encloses only fixed kinds of products. In the considered problem the lead time L is constant. As it was shown in many works, the analytical stochastic inventory control model is rather complex. As an alternative to the analytical approach, the authors have used simulation models realized in the simulation package Extend [3]. *The aim* of the given research is to construct the optimal ordering policy for the group of n products building a complex method, which uses the combination of analytical and simulation approaches.

We have to answer three basic questions: the quantity of each product ordering, the frequency of each product ordering, and the structure of the orders.

In the considered approach the authors propose to divide the process of problem solving into the following stages:

1) search of the problem investigation starting point:

a) Task1 – approximate solution of an inventory control deterministic problem (deterministic analytical approach);

b) Task2 – correction of the Task1 solution with the account of the losses from the possible goods deficit and random demand for the goods distributions (stochastic analytical approach);

2) Task3 – optimal solution of the inventory problem on the basis of the Task1 and Task2 results with the use of the simulation approach; note that the criteria of optimisation is minimum of the average total cost of the goods ordering, holding and loses from the deficit per a unit of time.

The matter of each task is considered in the given paper. The numerical results of problem solving are obtained accordingly in packages MS Excel (analytical calculations) and ExtendSim (simulation).

2. Approximate *S*earch of the Starting Point

Task1. In a real situation, different strategies of multiproduct ordering are used. In the simplest variant, all kinds of products can be included in each order. With the account of a possible considerable difference in demands for different goods, it may be more effective to use the strategy when each order encloses only fixed kinds of products. Let us denote the vector of the order as $Q = (Q_1, Q_2, ..., Q_n)$. In consideration of the second variant, we suppose that for the given order some products cannot be ordered, that means:

$(\forall i \notin Z) \rightarrow Q_i = 0$, where $Z \subseteq \{1,2,...,n\}$ is the set of the numbers of products, which can be included in the order. The last variant is the basic one in this paper.

Let D_i be the demand for the *i*-th product within a time unit (in our case one year). At the initial stage of the task solution we suppose that $D_i, i = 1,2,...,n$ are deterministic and equal to their average values. Denote by T time period between the moments of placing neighbouring orders; in considered task it is constant. We suppose that for each product there exists its own cycle of ordering and the duration of this cycle for *i*-th product is proportional to T with the coefficient of proportionality k_i. Time period T and coefficients k_i, $i = 1,2, ..., n$ are *control parameters* of the suggested model. We suppose that at the beginning of the considered process all *n* products are ordered. There might be a situation when at the moment of ordering $m \times T$ nothing is ordered, as it is illustrated in Table 1. Note the following: plus denotes that the *i*-th product can be included in the order; minus denotes that the *i*-th product cannot be included in the order.

Table 1. Rules for including the products in the order

i	k_i	Period of time													
		0	T	$2T$	$3T$	$4T$	$5T$	$6T$	$7T$	$8T$	$9T$	$10T$	$11T$	$12T$	…
1	2	+	–	+	–	+	–	+	–	+	–	+	–	+	…
2	3	+	–	–	+	–	–	+	–	–	+	–	–	+	…
3	2	+	–	+	–	+	–	+	–	+	–	+	–	+	…

In the considered problem we suppose that the following economic parameters are known:

– the ordering cost E_{OD} is a known function of the order $Q = (Q_1, \quad Q_2, ..., Q_n)$, i.e. $E_{OD} = f(Q)$;

– the holding cost of the *i*-th product is proportional to its quantity in the stock and the holding time with the coefficient of proportionality C_{H_i};

– the losses from the deficit of the *i*-th product are proportional to the quantity of its deficit with the coefficient of proportionality C_{SH_i}.

We denote l as the number of orders during a unit of time, i.e. $l = 1/T$. Let us suppose for simplicity that $1/T$ is an integer. Let the delivery at the nil-point of time has number 0, the delivery at time T has number 1, and the delivery at the time moment jT has number j. As we suppose, at the nil-point of time all products are ordered, then at the moment of time jT the product with number *i* has to be delivered if $[j/k_i] = j/k_i$, where $[a]$ is an integer part of number *a*.

Let $Q_j = (Q_{1,j}, \quad Q_{2,j}, ..., Q_{n,j})$ be the delivery with number j, where $Q_{i,j}$ is the quantity of the *i*-th product in this delivery:

$$Q_{i,j} = \begin{cases} D_i k_i T, & \text{if} \quad [j/k_i] = j/k_i; \\ 0, & \text{if} \quad [j/k_i] \neq j/k_i. \end{cases} \tag{1}$$

The holding cost of the *i*-th product during the year equals $C_{H_i}\frac{Q_i}{2}$ (see, for example [4], where $Q_i = D_i k_i T$, and the holding cost of *n* products E_H during the year is

$$E_H = \sum_{i=1}^{n} C_{H_i} \frac{Q_i}{2} = \frac{T}{2} \sum_{i=1}^{n} C_{H_i} k_i D_i \;. \tag{2}$$

The ordering cost E_{OD} is the sum of the orders forming the administrative expenditure E_{AD} and the products transportation expenditure E_{TR} during the year. If C_F is the expenditure for one order forming, then

$E_{AD} = l\, C_F = \frac{1}{T} C_F$ and

$$E_{OD} = E_{AD} + E_{TR} = \frac{1}{T} C_F + \sum_{j=0}^{l-1} C_{TR_j} \;, \tag{3}$$

where C_{TR_j} is the transportation expenditure of the *j*-th order.

From (2) and (3) we can find the total holding and ordering cost during a year:

$$E = E_H + E_{OD} = \frac{T}{2}\sum_{i=1}^{n} C_{H_i} k_i D_i + E_{AD} + \sum_{j=0}^{l-1} C_{TR_j} \ . \tag{4}$$

The cost of the i-th product transportation depends on the order quantity x, i.e. $C_{TR}^{(i)} = \varphi_i(x)$, and we suppose that $\varphi_i(0) = 0$. We shall consider the situation when the dependence of the transportation expenditure on the non-zero quantity of the i-th product is a linear one:

$$C_{TR}^{(i)} = \varphi_i(x) = \begin{cases} a_i + b_i x, & if \ x \neq 0; \\ 0, & if \ x = 0 \end{cases}, \tag{5}$$

where a_i and b_i are familiar coefficients.

Then the transportation expenditure of the j-th order C_{TR_j} can be calculated by the following formula:

$$C_{TR_j} = \sum_{i=1}^{n} (a_i + b_i Q_{i,j})\, y_{i,j} \ , \tag{6}$$

where $$y_{i,j} = \begin{cases} 1, & \text{if } Q_{i,j} \neq 0; \\ 0, & \text{if } Q_{i,j} = 0. \end{cases} \tag{7}$$

Taking into account (3) we have

$$E_{OD} = \frac{1}{T} C_F + \sum_{j=1}^{l}\sum_{i=1}^{n} (a_i + b_i Q_{i,j}) y_{i,j} \ . \tag{8}$$

The total holding and ordering cost E in this situation will be

$$E = \frac{T}{2}\sum_{i=1}^{n} C_{H_i} k_i D_i + \frac{1}{T} C_F + \sum_{j=1}^{l}\sum_{i=1}^{n} (a_i + b_i Q_{i,j}) y_{i,j} \ , \tag{9}$$

where $Q_{i,j}$ is defined by (1) and $y_{i,j}$ by (7).

In the previous consideration, the transportation expenditure was calculated separately for every order (see (6)-(8)) and it will be used for the Task3 solving.

Now we shall consider another approach when the transportation expenditure is calculated separately for each product during one year. This approach is more convenient for using it in Solver package, which is included in MS Excel.

Let $k_i T$ be the time period between the neighbouring orders of the i-th product, then $Q_i = D_i k_i T$ is the quantity of the i-th product in the corresponding delivery and the cost of this cargo transportation for one delivery is

$$a_i + b_i Q_i = a_i + b_i D_i k_i T \ . \tag{10}$$

The cost of the i-th product transportation for one year is

$$(a_i + b_i Q_i)\frac{1}{k_i T} = a_i \frac{1}{k_i T} + b_i D_i \ . \tag{11}$$

The total holding and ordering cost for all products during the year is

$$E^{year} = \frac{1}{2}\sum_{i=1}^{n} C_{H_i} Q_i + \frac{1}{T} C_F + \sum_{i=1}^{n} \left(\frac{a_i}{k_i T} + b_i D_i\right) == \frac{1}{2}\sum_{i=1}^{n} C_{H_i} D_i k_i T + \frac{1}{T} C_F + \sum_{i=1}^{n} \left(\frac{a_i}{k_i T} + b_i D_i\right). \tag{12}$$

The formula (12) determines the objective function of the model. So we can formulate the considered model:

$$E^{year} \to \min_{(k_i, i=1,\ldots,n; T)} \tag{13}$$

$k_i \geq 1$, $i = 1,2,\ldots,n$;

k_i is integer, $i = 1,2,...,n$;

$T > 0$,

where E^{year} is calculated by formula (12) and k_i, $i = 1,2,...,n$ and T are control parameters of the model.

Solving the optimisation problem (13) and finding the corresponding values of the control parameters, and using (6), it is possible to calculate the transportation cost for each order.

Task2. The next stage of considered problem solving is improving of the Task1 solution taking into account loses from the possible deficit. It is reasonable to increase the goods quantity until the expected losses from the deficit of the product unit are more then its holding cost. Let $C_{SH\,i}$ be the losses from the deficit of the *i*-th product unit, q_i be the quantity of the goods ordered for the period $\tau = k_i T$, and $P(D_i^{(\tau)} > q_i)$ be the probability that demand, $D_i^{(\tau)}$ for the *i*-th product during the period τ exceeds the quantity of goods in the stock. Then the expected losses from the deficit of the goods unit will be $P(D_i^{(\tau)} > q_i)C_{SH_i}$. The holding cost of the unit of the *i*-th product during the period τ is $C_{H_i} k_i T$.

From the condition $P(D_i^{(\tau)} > q_i)C_{SH_i} \ge C_{H_i} k_i T$ we have

$$P(D_i^{(\tau)} > q_i) \ge \frac{C_{H_i} k_i T}{C_{SH_i}}. \tag{14}$$

If the distribution of the *i*-th product demand for the period $\tau = k_i T$ is known, it is possible to find the goods quantity required for this period.

If the rest of the *i*-th product at the ordering moment is R_i and the lead time of the order is L, the order quantity for this product will be

$$Q_i = \begin{cases} q_i - (R_i - \overline{D}_i^{(L)}), & \text{if } R_i - \overline{D}_i^{(L)} > 0; \\ q_i, & \text{if } R_i - \overline{D}_i^{(L)} \le 0 \end{cases}, \tag{15}$$

where $\overline{D}_i^{(L)}$ is the average demand during the lead time.

3. Finding an Optimal Solution

Task3. The principal aim of the given task is to define the optimal variant of the control parameters T and k_i, $i = 1,2,...,n$, that minimizes the average total cost for the goods ordering, holding and losses from the deficit per year. In a general case we have to investigate the inventory control model using the set of values of parameters T and k_i, $i = 1,2,...,n$, which are located in the neighbourhood of the corresponding "starting point".

For the considered problem solving, the authors have used a simulation model realized in the package ExtendSim, which is the most powerful and flexible simulation tool for analysing, designing, and operating complex systems in the market. It enables the researcher to test the hypotheses without having to carry them out. ExtendSim has repeatedly proven its being capable of modelling large complex systems [5].

ExtendSim's design facilitates every phase of the simulation project, from creating, validating, and verifying the model, to the construction of the user interface, which allows others to analyse the system. ExtendSim products allow the researcher to build models of any size, to save changes for the existing models and to save the models themselves, print the model worksheets, and to use scripting for the automatic model. ExtendSim is a modular application. Its model is constructed with library-based iconic blocks. Each block describes a calculation or a step in the process.

The created simulation inventory model for multiproduct inventory control system for 5 products is shown on Figure 1. The model consists of four parts presented correspondingly in figures named as a, b, c, d.

Let us consider the main blocks of the simulation subschema shown on Figure 1a. Block #1 is the transact creation block; block #2 is the orders counter; block#3 is the delivery simulation block; block #4 produces the order allocation by the goods type; blocks #5 – #9 are the warehouse simulation custom blocks; separate exit for goods and deficit simulation transacts are made in block #10. The purposes of blocks shown on Figures 1b-1d are given in captions.

Using the created simulation model, we can find an average total cost for the goods ordering, holding and losses from the deficit per year in inventory system for different variants of the control parameters (coefficients of proportionality k_i, $i = 1,2,...,n$ and the time period between the neighbouring orders T). The optimal solution of the problem is selected from the set of considered variants using the optimisation criteria – the minimum of the average total cost (see example in the next section). As a starting point for k_i, $i = 1,2,...,n$ and T we shall take the results of the task (13); and the order quantity for each product will be calculated by using formula (15).

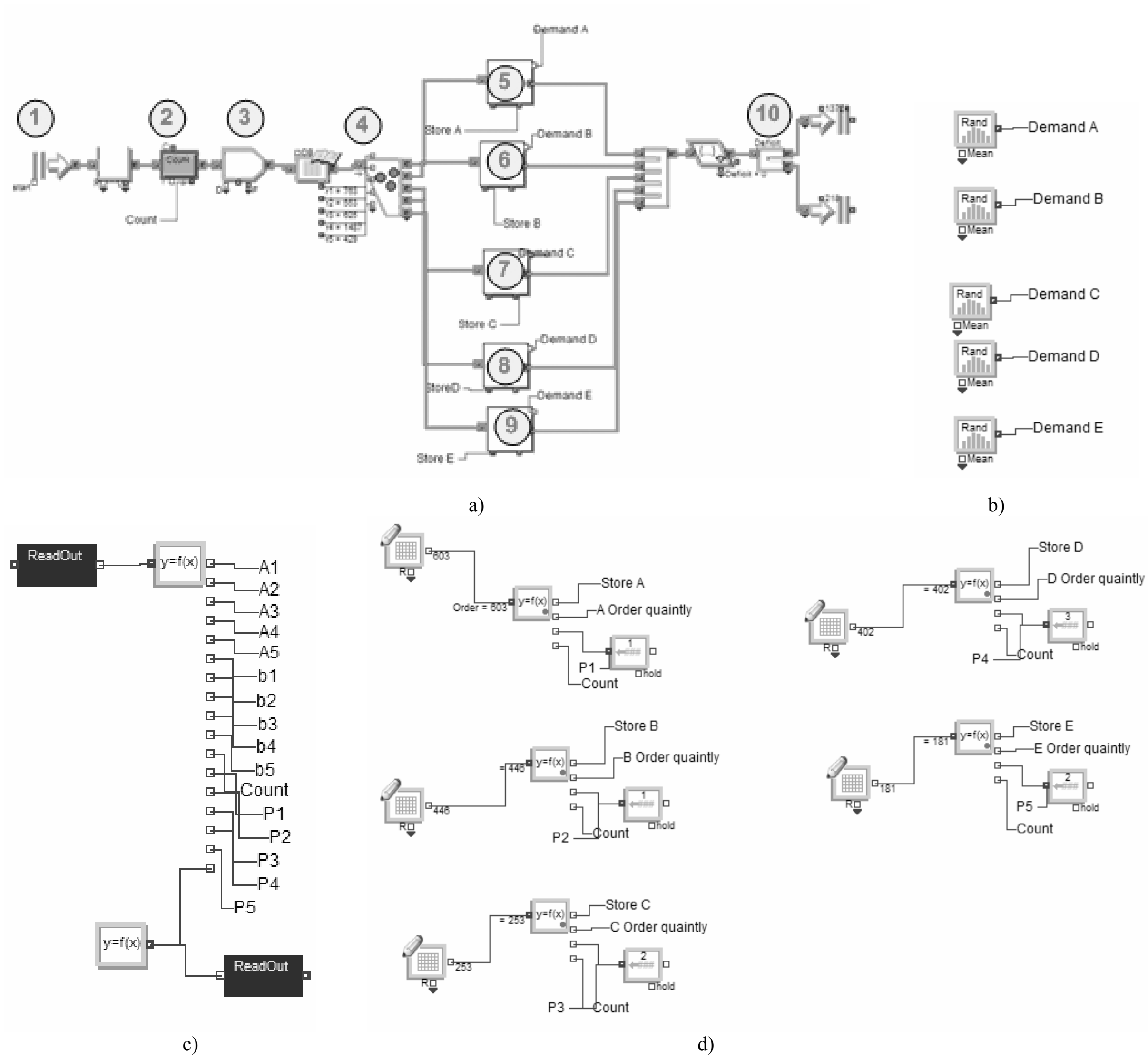

Figure 1. Inventory control simulation model realized in ExtendSim
a – main part; b – demand simulation blocks; c – cost calculation blocks; d – order quantity calculation blocks

4. Example

Let us consider an inventory control system with 5 different products. Demands for these products are random and have normal distribution with the known parameters as the mean μ_i and as the standard deviation σ_i (see Table 2). Shortage C_{SH_i} and holding C_{H_i} costs, coefficients a_i and b_i for the transportation expenditure calculation are given in Table 2, too. Lead time L is constant and equals 7 days. Administrative expenditure for one order C_F is equal to 210 EUR.

Table 2. Basic data

Parameters	Numbers of product, *i*				
	1	2	3	4	5
μ_i , units/day	530,61	382,26	437,18	1073,0	331,53
σ_i , units/day	148,24	105,12	92,16	206,14	148,15
C_{SH_i} , EUR/unit	10,50	9,80	5,50	9,10	6,00
C_{H_i} , EUR/year	5,50	6,50	9,45	4,00	4,50
a_i , EUR	200	300	500	900	300
b_i , EUR/unit	0,80	1,20	0,95	1,30	0,90

The numerical results of Task1 solving are obtained in the package MS Excel and are presented in Table 3. Note that in considered case time period between neighbouring orders is $T = [62{,}08] = 62$ days.

Table 3. Results of Task1

Parameters	Numbers of product, *i*				
	1	2	3	4	5
k_i	1	1	2	3	2
$k_i \cdot T$, days	62	62	124	186	124

The values of left and right parts expressions of inequality (14) for every type of the goods and the corresponding values of their quantities q_i are given in Table 4.

Table 4. Results of Task2

Parameters	Numbers of product, *i*				
	1	2	3	4	5
$\frac{C_{H_i} k_i T}{C_{SH_i}}$	0,0089	0,011	0,021	0,022	0,254
$P(D_i^{(\tau)} > q_i)$	0,9911	0,989	0,979	0,978	0,746
q_i , units	881,5	622,8	624,9	1487	429,0

Using results of Task1 and Task2 presented in Tables 3 and 4, we can form Table 5, which encloses basic data for Task3.

Table 5. Basic data for Task3

Parameters	Numbers of product, *i*				
	1	2	3	4	5
k_i	1	1	2	3	2
$k_i T$, days	62	62	124	186	124
q_i , units/ cycle	882	623	625	1487	429

Let us consider the solution of Task3 using the created simulation model. The period of simulation is one year and the number of realization is 100.

An example of the inventory control process simulation (one realization) is shown on Figure 2. Note, that each figure presents up three plots. So, two figures describe one realization. As it can be easily observed, five products have different multiple cycles of ordering. There are cases of deficit for three products.

In the given example we have investigated different variants $T_0, T_1, ..., T_g$ of control parameter T , starting from point $T = 62$ days and using the initial data from Table 5, and we have suggested that the control parameters $k_i, i = 1,2,...5$ are permanent in all the variants considered herein. The optimal decision T^{opt} is selected from the set of $\{T_0, T_1, ..., T_g\}$ using the optimisation criteria – the minimum of the average total cost for ordering the goods, holding and losses from the deficit per a unit of time, i.e.

$$\overline{E^{year}}(T^{opt}) = \min_{T_i, i=0,1,..,g} \left[\overline{E^{year}}(T_i) \right], \tag{16}$$

where $\overline{E^{year}}(T_i)$ is the average total cost (expenses) for ordering the goods, holding and losses from the deficit per a year for the variant with time period T_i .

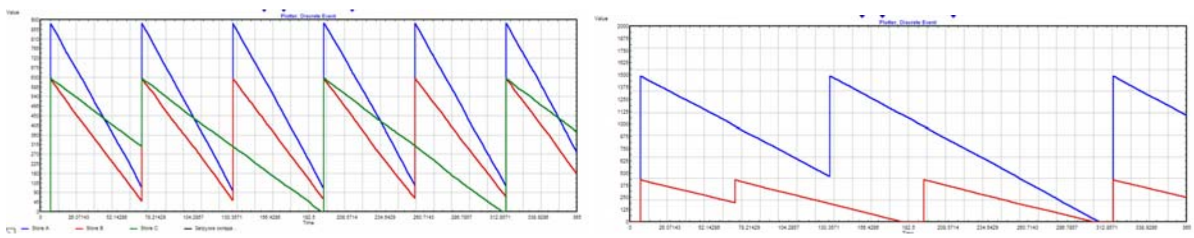

Figure 2. Example of simulation of the inventory control process

The results of the simulation are shown in Table 6. For the given steps of the control parameter T changing, the best result is achieved at point $T = 61$ units, where for 100 realizations the average total cost of one year period equals 27880 EUR.

Table 6. Average expenses for the goods holding, ordering and losses from the deficit per a year

Time cycle T, day	59	60	61
Total cost $\overline{YC}(T)$, EUR	29618	28620	**27880**
Time cycle T, day	62	63	64
Total cost $\overline{YC}(T)$, EUR	28433	28982	30054

Conclusions

1. The considered model is suggested for the systems with a fixed moment of placing orders. The principal aim of the task is to define the structure of each multi-product order and the moments of orders to achieve the minimum expenses for the goods holding, ordering and losses from the deficit per a time unit. Each order may enclose only fixed kinds of products; we assume that for each product there exists a cycle of ordering; and the coefficient of the ordering ratio must be determined for each kind of products, this coefficient is also a control parameter.

2. The process of the task solving is divided into two stages:

- search of the starting point (an approximate solution of the inventory control deterministic problem and the correction of the received solution with the account of losses from the possible goods deficit and the random demand distributions);
- finding an optimal solution of the inventory problem using the simulation approach.

3. Further guidelines of the current research are the following: to consider the case with random lead time for the goods; to investigate the multi-product inventory control problem with certain constraints.

References

1. Chopra, S. and P. Meindl. *Supply Chain Management*. London: Prentice Hall, 2001.
2. Ross, Sh. M. *Applied Probability Models with Optimization Applications*. New York: Dover Publications, INC., 1992.
3. Kopytov, E., Greenglaz, L., Muravjov, A. and E. Puzinkevich. Modelling of Two Strategies in Inventory Control System with Random Lead Time and Demand. In: *Computer Modelling & New Technologies*, Vol. 11(1), 2007, pp. 21-30. (Riga, Transport and Telecommunication Institute)
4. Kopytov, E., Greenglaz, L. On a task of optimal inventory control. In: *Proceeding of XXIV International Seminar on Stability Problems for Stochastic Models, September 9-17, 2004. Jurmala, Latvia.* Riga: Transport and Telecommunication Institute, 2004, pp. 247-252.
5. Krahl, D. ExtendSim 7. In: *Proceedings of the 39th conference on winter simulation: 40 years! December 09-12, 2007 / Ed. by S. G. Henderson, B. Biller, M.-H. Hsieh, J. Shortle, J. D. Tew, and R.R. Barton.* Washington D.C., 2007, pp. 226-232. Available – http://www.informs-sim.org/wsc07papers/024.pdf [accessed April 21, 2008].