ON A COPULA FOR A RELIABILITY FUNCTION OF A MULTICOMPONENT SYSTEM

Alexander Andronov

Riga Technical University
Kalku str. 1, Riga, Latvia, LV-1658
E-mail: Aleksandrs.Andronovs@rtu.lv

A considered reliability system consists from n identical elements. Each element can be available or failed. A failure rate of element equals to λ / i, where i is a number of available elements. Time till a failure of an element has exponential distribution. Independence property has place too. Therefore total failure rate is the constant λ, until at last one element is available. Initially all elements are available. The main aim of the paper is to determine a joint distribution of the available time for all elements, as so as a corresponding copula. It allows us to generalize gotten results on unexponential case.

Keywords: m*ulti-component system, reliability function, copula*

1. Introduction

Usually described in the literature reliability systems are considered under supposition that system's elements fail independently [3, 5]. However, often a case takes place when failure of some elements increases a load on worked elements, so element available times are dependent random variable [6, 7]. In connection with that chosen problem of the corresponding multidimensional distribution for fitting the given statistical data is actual one.

In econometric often such problem's solution uses so-called *copulas* [1, 4, 10]. Joint distribution function $C(u_1, u_2, ..., u_n) = P\{U_1 \le u_1, U_2 \le u_2, ..., U_n \le u_n\}$ is called *copula* if the marginal distributions of all components $U_1, U_2, ..., U_n$ are uniform on [0, 1]. The following fact is basic [9]: any multivariate continues distribution function $G(x_1, x_2, ..., x_n) = P\{X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\}$ can be presented uniquely via cumulative distribution function of its component $F_i(x_i) = P\{X_i \le x_i\}$ by corresponding copula C:

$$G(x_1, x_2, ..., x_n) = C\big(F_1(x_1), F_2(x_2), ..., F_n(x_n)\big).$$

An aim of current paper is using copula-based approach to description of reliability function of system that elements have dependent available times. We consider concrete system of such a sort and derive corresponding copula $C(u_1, u_2, ..., u_n)$. Then we can use this copula with various marginal distributions of components getting a distribution family for fitting aim.

Let us describe a considered system. One consists from n elements. Each element can be in two states: available and failed. A failure rate for each element is λ/i, where i is a number of available elements, with that corresponding time till failure has exponential distribution. Independence property has place too. Therefore total failure rate is constant λ, until at least one element is available.

We would like to answer on the following questions:

1. What is distribution of time moment T when the last element fails?
2. What is marginal distribution of available time X for fixed element?
3. What is joint distribution of available times T for two and more fixed elements?
4. What is corresponding copulas?

To get answers on two first questions, we take in mind the following. 1) Failure flow is a Poisson's process with intensity λ, till a time moment when the last element fails. 2) The time moment of the i–th failure has Erlang's distribution with parameters λ and n [8]. 3) Each such moment is a failure time moment of some element. The fixed element fails as i-th in succession with probability $1/n$.

Distribution of time moment T when the last element fails. Here we have Erlang's distribution [8] with parameters λ and n:

$$R\{T \le t\} = 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}, t > 0. \qquad (1)$$

The marginal probability density function for available time X of fixed element. Here we have the following expression:

$$p_\lambda(x) = \frac{1}{n}\sum_{i=1}^{n} \lambda \frac{(\lambda x)^{i-1}}{(i-1)!} e^{-\lambda x}, x \ge 0. \tag{2}$$

A corresponding distribution function is

$$\begin{aligned} P_\lambda(x) = P\{X \le x\} &= \int_0^x p(z)dz = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \sum_{j=0}^{i-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}\right) = \\ &= 1 - \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}\frac{(\lambda x)^j}{j!}e^{-\lambda x} = 1 - \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\frac{(\lambda x)^j}{j!}e^{-\lambda x} = \\ &= 1 - (\lambda x)^{n-1}\frac{1}{(n-1)!}e^{-\lambda x} - \left(1 - \frac{\lambda x}{n}\right)\sum_{i=0}^{n-2}\frac{(\lambda x)^i}{i!}e^{-\lambda x}, x \ge 0. \end{aligned} \tag{3}$$

2. Two-dimensional case

The joint distribution for available times (X, Y) for two fixed elements. Marginal probability's density function

$$g(x,y) = \frac{\partial^2}{\partial x \partial y} P\{X \le x, Y \le y\} = \begin{cases} \frac{1}{nx}\left\{\sum_{i=2}^{n}\frac{i-1}{n-1}\lambda\frac{(\lambda x)^{i-1}}{(i-1)!}e^{-\lambda x}\right\} & if\ x \ge y, \\ \frac{1}{ny}\left\{\sum_{i=2}^{n}\frac{i-1}{n-1}\lambda\frac{(\lambda y)^{i-1}}{(i-1)!}e^{-\lambda y}\right\} & if\ y \ge x. \end{cases} \tag{4}$$

Here we use the following facts. 1) If the *i-th* event of the Poisson's process occurs at the time moment x then previous i - 1 events are distributed uniformly and independently on $(0, x)$. 2) If the fixed element fails as *i*-th in succession that an another element fails earlier with probability $(i - 1)/(n - 1)$.

After simplification we get

$$g(x,y) = \begin{cases} \frac{\lambda}{n}\left\{\sum_{i=2}^{n}\frac{1}{n-1}\lambda\frac{(\lambda x)^{i-2}}{(i-2)!}e^{-\lambda x}\right\} = \frac{\lambda}{n(n-1)}\left\{\sum_{i=0}^{n-2}\lambda\frac{(\lambda x)^{i}}{i!}e^{-\lambda x}\right\} & if\ x \ge y, \\ \frac{\lambda}{n}\left\{\sum_{i=2}^{n}\frac{1}{n-1}\lambda\frac{(\lambda y)^{i-2}}{(i-2)!}e^{-\lambda y}\right\} = \frac{\lambda}{n(n-1)}\left\{\sum_{i=0}^{n-2}\lambda\frac{(\lambda y)^{i}}{i!}e^{-\lambda y}\right\} & if\ y \ge x. \end{cases} \tag{5}$$

Now we wish to find the distribution function $G(x, y)$. The corresponding proof is gotten in the Appendix. Finally for $x < y$ we have:

$$\begin{aligned} G(x,y) &= 1 - \frac{1}{n(n-1)}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}(n-j)(n+j-1) + \\ &+ \frac{\lambda x}{n(n-1)}\sum_{j=0}^{n-2}\frac{\lambda^j}{j!}(n-j-1)\{x^j e^{-\lambda x} - y^j e^{-\lambda y}\}. \end{aligned}$$

An equivalent representation is

$$\begin{aligned} G(x,y) &= 1 - \sum_{j=0}^{n-1}\frac{1}{j!}(\lambda x)^j e^{-\lambda x} + \\ &+ \lambda x\frac{1}{n}\sum_{j=0}^{n-2}\left(x^j e^{-\lambda x} - y^j e^{-\lambda y}\right)\frac{\lambda^j}{j!} + xy\frac{\lambda^2}{n(n-1)}\sum_{j=0}^{n-3}\frac{1}{j!}(\lambda y)^j e^{-\lambda y}. \end{aligned} \tag{6}$$

Till the end of the current section, we will consider a two-dimensional case for $n = 2$. For x<y

$$G(x,y) = 1 - e^{-\lambda x} - \lambda x e^{-\lambda x} + \frac{1}{2}\lambda x\left(e^{-\lambda x} - e^{-\lambda y}\right) = \\ = 1 - e^{-\lambda x} - \frac{1}{2}\lambda x e^{-\lambda x} - \frac{1}{2}\lambda x e^{-\lambda y}. \tag{7}$$

This formula can be obtained directly by the following reasoning. Any random variable from X and Y can take minimal value. Therefore for x < y:

$$G(x,y) = \frac{1}{2}\int_0^x \lambda e^{-\lambda z}\left(1 - e^{-\lambda(y-z)}\right)dz + \frac{1}{2}\int_0^x \lambda e^{-\lambda u}\left(1 - e^{-\lambda(x-u)}\right)du =$$

$$= \frac{1}{2}\left(1 - e^{-\lambda x}\right) - \frac{1}{2}\int_0^x \lambda e^{-\lambda z} e^{-\lambda(y-z)}dz + \frac{1}{2}\left(1 - e^{-\lambda x}\right) - \frac{1}{2}\int_0^x \lambda e^{-\lambda u} e^{-\lambda(x-u)}dz$$

$$= \left(1 - e^{-\lambda x}\right) - \frac{1}{2}\int_0^x \lambda e^{-\lambda y}dz - \frac{1}{2}\int_0^x \lambda e^{-\lambda x}dz =$$

$$= \left(1 - e^{-\lambda x}\right) - \frac{1}{2}\lambda x e^{-\lambda y} - \frac{1}{2}\lambda x e^{-\lambda x}.$$

Note the marginal distribution function can be gotten setting $y = \infty$:

$$P_\lambda(x) = P\{X \le x\} = 1 - \left(e^{-\lambda x} + \frac{1}{2}\lambda x e^{-\lambda x}\right), x \ge 0, \tag{8}$$

that corresponds to (3) for $n = 2$.

What is the corresponding copula? Let $q(p) > 0$ be the root of the equation

$$p = 1 - \left(e^{-x} + \frac{1}{2}x e^{-x}\right),\ 0 < p < 1. \tag{9}$$

Obviously it is the p-quantile of the distribution (7) for $\lambda = 1$:

$$P_1(q(p)) = 1 - \left(e^{-q(p)} + \frac{1}{2}q(p)e^{-q(p)}\right) = p,\ 0 < p < 1.$$

Substitution $p = P_\lambda(x)$ gives

$$P_1(q(P_\lambda(x))) = 1 - \left(e^{-q(P_\lambda(x))} + \frac{1}{2}q(P_\lambda(x))e^{-q(P_\lambda(x))}\right) = P_\lambda(x). \tag{10}$$

By a comparison with (8) we get

$$\lambda x = q\left(P_\lambda(x)\right).$$

Further,

$$e^{-\lambda y} = e^{-q\left(P_\lambda(y)\right)}.$$

Now we are able to represent (7) for $x < y$ in the form

$$G(x,y) = 1 - e^{-\lambda x} - \frac{1}{2}\lambda x e^{-\lambda x} - \frac{\lambda x}{2}e^{-\lambda y} = \\ = P_\lambda(x) - \frac{1}{2}q(P_\lambda(x))e^{-q(P_\lambda(y))}. \tag{11}$$

On the other hand the formula (10) gives equality

$$e^{-q(P_\lambda(y))} = (1-P_\lambda(y))\left(1+\frac{1}{2}q(P_\lambda(y))\right)^{-1}.$$

Therefore an equivalent representation of (11) for $x < y$ is

$$G(x,y) = 1 - e^{-\lambda x} - \frac{1}{2}\lambda x e^{-\lambda x} - \frac{\lambda x}{2}e^{-\lambda y} =$$
$$= P_\lambda(x) - \frac{1}{2}q(P_\lambda(x))(1-P_\lambda(y))\left(1+\frac{1}{2}q(P_\lambda(y))\right).$$

Therefore a copula of interest is as follows:

$$C(u,v) = u - \frac{1}{2}q(u)e^{-q(v)} =$$
$$= u - \frac{1}{2}q(u)(1-v)\left(1+\frac{1}{2}q(v)\right), \;\; u < v. \tag{12}$$

Now the joint distribution function $G(x, y)$ for $x < y$ can be represented in the form

$$G(x,y) = C(P_\lambda(x), P_\lambda(y)),$$

where $P_\lambda(x)$ and $P_\lambda(y)$ are calculated by (8).

3. Multi-Dimensional Case

Joint density distribution for $X_1, X_2, ..., X_n$ obviously is

$$g(x_1,x_2,...,x_n) = \frac{1}{n}x_{i*}^{-(n-1)}\frac{1}{(n-1)!}\lambda(\lambda x_{i*})^{n-1}e^{-\lambda x_{i*}} = \frac{1}{n!}\lambda^n e^{-\lambda x_i}, \forall x_i \ge 0, \tag{13}$$

where $x_{i*} = \max\{x_1, x_2, ..., x_n\}$.

Let us verify the normalization condition. For $i* = 1$ we have

$$\int_0^\infty\int_0^z ... \int_0^z g(z,x_2,...,x_n)dx_x...dx_n dz = \int_0^\infty z^{n-1}\frac{1}{n!}\lambda^n e^{-\lambda z}dz = \frac{1}{n}.$$

This result can be multiplying by n because full integral contains possibilities that maximal component can take any place from n ones ($i* = 1, 2,... ,n$). So the normalization condition is fulfilled.

One can get that joint density distribution for $X_1, X_2, ..., X_k$, $1 < k \le n$, has the following form:

$$g(x_1,x_2,...,x_k) = \frac{1}{n}x_{i*}^{-(k-1)}\sum_{i=k}^{n}\frac{(i-1)(i-2)...(i-k+1)}{(n-1)(n-2)...(n-k+1)}\frac{1}{(i-1)!}\lambda(\lambda x_{i*})^{i-1}e^{-\lambda x_{i*}}, \tag{14}$$

where $x_{i*} = \max\{x_1, x_2, ..., x_k\}$.

The last expression is valid for $k = 1$ too, if we set $(i - 1)(i - 2)\ldots i = 1$, and $(n - 1)(n - 2)\ldots n = 1$.

Now we intend to get an expression for joint distribution function $G(x_1,x_2,...,x_n) = P\{X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\}$. At first we consider a case $x_1 < x_2 < ... < x_n$. The event $\{X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\}$ consists from n disjoint events, when the random variable X_i, $i = 1, 2,\ldots, n$ takes the maximal value. If we denote this maximal value by z, then

$$G(x_1, x_2, ..., x_n) = \int_0^{x_1}\int_0^{z}...\int_0^{z} g(z, v_2, ..., v_n)dv_n...dv_2dz +$$
$$+\sum_{i=2}^{n}\{\int_0^{x_1}\int_0^{z}...\int_0^{z} g(v_1, v_2, ..., v_{i-1}, z, v_{i+1}, ..., v_n)dv_n dv_{n-1}...dv_{i+1}dv_{i-1}...dv_1 dz +$$
$$+\int_{x_1}^{x_2}\int_0^{x_1}\int_0^{z}...\int_0^{z} g(v_1, v_2, ..., v_{i-1}, z, v_{i+1}, ..., v_n)dv_n dv_{n-1}...dv_{i+1}dv_{i-1}...dv_1 dz +$$
$$+\int_{x_2}^{x_3}\int_0^{x_1}\int_0^{x_2}\int_0^{z}...\int_0^{z} g(v_1, v_2, ..., v_{i-1}, z, v_{i+1}, ..., v_n)dv_n dv_{n-1}...dv_{i+1}dv_{i-1}...dv_1 dz + ...$$
$$... + \int_{x_{i-1}}^{x_i}\int_0^{x_1}...\int_0^{x_{i-1}}\int_0^{z}...\int_0^{z} g(v_1, v_2, ..., v_{i-1}, z, v_{i+1}, ..., v_n)dv_n dv_{n-1}...dv_{i+1}dv_{i-1}...dv_1 dz\}.$$

A substitution (13) in the integral gives the following expression:

$$G(x_1, x_2, ..., x_n) = \frac{1}{n}\left(1 - \sum_{j=0}^{n-1}\frac{1}{j!}(\lambda x_1)^j e^{-\lambda x_1}\right) + \sum_{i=2}^{n}\{\frac{1}{n}\left(1 - \sum_{j=0}^{n-1}\frac{1}{j!}(\lambda x_1)^j e^{-\lambda x_1}\right) +$$
$$+\int_{x_1}^{x_2}\int_0^{x_1} z^{n-2}\frac{\lambda^n}{n!}e^{-\lambda z}dv_1 dz + \int_{x_2}^{x_3}\int_0^{x_1}\int_0^{x_2} z^{n-3}\frac{\lambda^n}{n!}e^{-\lambda z}dv_2 dv_1 dz + ... +$$
$$+\int_{x_{i-1}}^{x_i}\int_0^{x_1}...\int_0^{x_{i-1}} z^{n-i}\frac{\lambda^n}{n!}e^{-\lambda z}dv_{i-i}...dv_1 dz\} =$$
$$= 1 - \sum_{j=0}^{n-1}\frac{1}{j!}(\lambda x_1)^j e^{-\lambda x_1} + \sum_{i=2}^{n}\{x_1\frac{\lambda}{n(n-1)}\left[\sum_{j=0}^{n-2}\left(x_1^j e^{-\lambda x_1} - x_2^j e^{-\lambda x_2}\right)\frac{\lambda^j}{j!}\right] +$$
$$+ x_1 x_2 \frac{\lambda^2}{n(n-1)(n-2)}\left[\sum_{j=0}^{n-3}\left(x_2^j e^{-\lambda x_2} - x_3^j e^{-\lambda x_3}\right)\frac{\lambda^j}{j!}\right] + ... +$$
$$+ x_1 x_2 ... x_{i-1}\frac{\lambda^{i-1}}{n(n-1)...(n-i+1)}\left[\sum_{j=0}^{n-i}\left(x_{i-1}^j e^{-\lambda x_{i-1}} - x_i^j e^{-\lambda x_i}\right)\frac{\lambda^j}{j!}\right]\}.$$

Final formula has the following view:

$$G(x_1, x_2, ..., x_n) = 1 - \sum_{j=0}^{n-1}\frac{1}{j!}(\lambda x_1)^j e^{-\lambda x_1} +$$
$$+\sum_{i=2}^{n} x_1 x_2 ....x_{i-1}\frac{\lambda^{i-1}}{n(n-1)...(n-i+1)}(n-i+1)\sum_{j=0}^{n-i}\left(x_{i-1}^j e^{-\lambda x_{i-1}} - x_i^j e^{-\lambda x_i}\right)\frac{\lambda^j}{j!}. \quad (15)$$

Now we consider a case when the sequence $x_1, x_2, ..., x_n$ is not ordered. Let $\pi = (\pi(1), \pi(2), ..., \pi(n))$ be a permutation of {1, 2, …, *n*}. We consider $\pi(i)$ as a number of the element $x_{\pi(i)}$ that take *i*-th place in the ordered sequence $x^{(1)} < x^{(2)} < ... < x^{(n)}$: $x_{\pi(i)} = x^{(i).}$. Then previous formula takes place if we change the index *i* by $\pi(i)$. So we have

$$G(x_1, x_2, ..., x_n) = 1 - \sum_{j=0}^{n-1}\frac{1}{j!}\left(\lambda x_{\pi(1)}\right)^j e^{-\lambda x_{\pi(1)}} +$$
$$+\sum_{i=2}^{n} x_{\pi(1)}x_{\pi(2)}....x_{\pi(i-1)}\frac{\lambda^{i-1}}{n(n-1)...(n-i+1)}(n-i+1)\sum_{j=0}^{n-i}\left(x_{\pi(i-1)}^j e^{-\lambda x_{\pi(i-1)}} - x_{\pi(i)}^j)e^{-\lambda x_{\pi(i)}}\right)\frac{\lambda^j}{j!}.$$

In fact to calculate the distribution function $G(x_1, x_2, ..., x_n)$ for arbitrary order of $x_1, x_2, ..., x_n$, we must range this sequence, get ordered one $x^{(1)} < x^{(2)} < ... < x^{(n)}$, and use formula (15) for $x_i = x^{(i)}$.

Our last aim is to find an expression for a copula corresponding to the joint distribution (15).
Let $q(p) > 0$ be the root of the equation

$$p = 1 - \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\frac{1}{j!}x^j e^{-x}, 0 < p < 1. \tag{16}$$

Obviously it is the p-quantile of the distribution (3) for $\lambda = 1$:

$$P_1(q(p)) = 1 - \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\frac{1}{j!}q(p)^j e^{-q(p)} = p,\ \ 0 < p < 1.$$

Substitution $p = P_\lambda(x)$ gives

$$P_1(q(P_\lambda(x))) = 1 - \frac{1}{n}\sum_{j=0}^{n-1}(n-j)\frac{1}{j!}q(P_\lambda(x)))^j e^{-q(P_\lambda(x))} = P_\lambda(x),\ \ x > 0.$$

By a comparison with (3) we get

$$\lambda x = q(P_\lambda(x)).$$

It allows us to represent (15) for $x_1 < x_2 < ... < x_n$ by such way:

$$\begin{aligned} G(x_1, x_2, ..., x_n) &= 1 - \sum_{j=0}^{n-1}\frac{1}{j!}q(P_\lambda(x_1))^j e^{-q(P_\lambda(x_1))} + \\ &+ \sum_{i=2}^{n} q(P_\lambda(x_1))...q(P_\lambda(x_{i-1}))\frac{1}{n(n-1)...(n-i+2)}\times \\ &\times \sum_{j=0}^{n-i}\left(q(P_\lambda(x_{i-1}))^j e^{-q(P_\lambda(x_{i-1}))} - q(P_\lambda(x_i))^j e^{-q(P_\lambda(x_i))}\right)\frac{1}{j!}. \end{aligned} \tag{17}$$

Therefore a copula of interest for $0 < u_1 < u_2 < ... < u_n < 1$ has the following form:

$$\begin{aligned} C(u_1, u_2, ..., u_n) &= 1 - \sum_{j=0}^{n-1}\frac{1}{j!}q(u_1)^j e^{-q(u_1)} + \\ &+ \sum_{i=2}^{n} q(u_1)...q(u_{i-1})\frac{1}{n(n-1)...(n-i+2)}\times \\ &\times \sum_{j=0}^{n-i}\left(q(u_{i-1})^j e^{-q(u_{i-1})} - q(u_i)^j e^{-q(u_i)}\right)\frac{1}{j!}. \end{aligned} \tag{18}$$

Using this formula one can represent the joint distribution (15) for $x_1 < x_2 < ... < x_n$ in the form

$$G(x_1, x_2, .., x_n) = C(P_\lambda(x_1), P_\lambda(x_2), .., P_\lambda(x_n)),$$

where $P_\lambda(x_i)$ are calculated by (3).

Now we are able to generalize our results and consider a family of reliability functions of the system for $x_1 < x_2 < ... < x_n$ as

$$R(x_1, x_2, .., x_n) = C(F(x_1), F(x_2), .., F(x_n)),$$

where $F(x)$ is an arbitrary reliability function of elements.

Note a case of unordered sequence $x_1, x_2, ..., x_n$ is considered as earlier.

Example

We consider our basic model for the following data: $n = 4$, $\lambda = 2$. Therefore for initial failure rate $\lambda/n = 0.5$, a fixed element has the following values of the mean μ and the standard deviation σ of time till a failure: $\mu = \sigma = n/\lambda = 2$. The table contains values of the joint distribution function (15) for different values of argument $x = (x_1 \quad x_2 \quad x_3 \quad x_4)^T$.

Now we show how the same copula (18) can be used for another marginal distributions $F(x)$. As the last uniform distribution $U(x)$ and lognormal distribution $L(x)$ are chosen:

$$U(x) = \begin{cases} 0 \ \ if \ x < 0, \\ \dfrac{x}{2\mu} \ \ if \ 0 < x < 2\mu, \\ 1 \ \ if \ x > 2\mu, \end{cases} \qquad L(x) = \begin{cases} 0 \ \ if \ x < 0, \\ \Phi\left(\dfrac{\ln(x) - a}{s}\right) \ \ if \ x > 0, \end{cases}$$

where $\Phi(z)$ is the cumulative distribution function of the standard normal distribution,
$a, s > 0$ are parameters of lognormal distribution.
The parameters of both distribution $U(x)$ and $L(x)$ were chosen in such a way that expectation μ coincide for all three distributions. For the lognormai distribution standard deviation σ coincides too. Let us remember that μ and σ of the lognormal distribution are calculated with respect to formulas

$$\mu = \exp\left(a + \frac{1}{2}s^2\right), \quad \sigma^2 = \left(\exp(s^2) - 1\right)\exp\left(2a + s^2\right).$$

For our numerical data $a = 0.3464$ and $s = 0.8326$.

Corresponding values of the joint distribution functions $GU(x)$ and $GL(x)$ are represented in the Table too. A comparing of all three distributions shows that the considered copula (18) generates different distributions. It allows us to suggest using the considered model for a description of complex reliability systems. An estimation of unknown parameters can be performed, in particular, analogously to the paper [2].

Conclusions

In the paper new family of multivariate distribution functions for nonnegative random variables is suggested. All distributions have the same copula and differ one from other by marginal distributions. The last allows choosing a multivariate distribution that in the best way fits given data of system reliability.

Table 1. Values of the joint distribution functions

x = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$	$\begin{pmatrix} 0.5 \\ 1 \\ 1.5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0.5 \\ 1.5 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1.5 \\ 3 \\ 3.5 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 3.5 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1.5 \\ 2.5 \\ 3.5 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \\ 3.5 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 3.5 \\ 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 6.5 \\ 6.5 \\ 6.5 \end{pmatrix}$
G(x)	0.122	0.185	0.363	0.415	0.567	0.621	0.767	0.849	0.912	0.999
GU(x)	0.021	0.047	0.107	0.136	0.159	0.264	0.400	0.473	0.690	1.000
GL(x)	0.032	0.060	0.198	0.236	0.341	0.410	0.534	0.589	0.677	0.904

Appendix. *Proof of formula (6).* We give two proofs. The first proof uses the formula (15) for multivariate case; the second proof is direct one. It allows us to satisfy one that the both expressions (6) and (15) are true.

From (15) for $x < y$ we have:

$$G(x_1, x_2, \infty, \ldots, \infty) = 1 - \sum_{j=0}^{n-1} \frac{1}{j!} (\lambda x_1)^j e^{-\lambda x_1} +$$

$$+ x_1 \frac{\lambda^{2-1}}{n(n-1)\ldots(n-2+1)} (n-2+1) \sum_{j=0}^{n-2} \left(x_{2-1}^j e^{-\lambda x_{2-1}} - x_2^j e^{-\lambda x_2} \right) \frac{\lambda^j}{j!} +$$

$$+ x_1 x_2 \frac{\lambda^{3-1}}{n(n-1)\ldots(n-3+1)} (n-3+1) \sum_{j=0}^{n-3} \left(x_{3-1}^j e^{-\lambda x_{3-1}} - 0 \right) \frac{\lambda^j}{j!} =$$

$$= 1 - \sum_{j=0}^{n-1} \frac{1}{j!} (\lambda x_1)^j e^{-\lambda x_1} + \lambda x_1 \frac{1}{n} \sum_{j=0}^{n-2} \left(x_1^j e^{-\lambda x_1} - x_2^j e^{-\lambda x_2} \right) \frac{\lambda^j}{j!} + x_1 x_2 \frac{\lambda^2}{n(n-1)} \sum_{j=0}^{n-3} \frac{1}{j!} (\lambda x_2)^j e^{-\lambda x_2}.$$

The gotten expression coincides with (6).

Now let us consider direct proof for $x < y$. A current value for the random variable X we denote as z and for the random variable Y as u. It is necessary to consider two possible cases: $X = z < Y = u$ and $Y = u < X = z$. We have:

$$G(x, y) = \int_0^x \int_0^y g(z,u) du dz = \int_0^x \left\{ \int_0^z g(z,u) du + \int_z^y g(z,u) du \right\} dz =$$

$$= \int_0^x \left\{ \int_0^z \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda z)^i}{i!} e^{-\lambda z} \right\} du + \int_z^y \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} du \right\} dz =$$

$$= \int_0^x \frac{\lambda z}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda z)^i}{i!} e^{-\lambda z} \right\} dz + \int_0^x \left\{ \int_0^u \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} dz \right\} du +$$

$$+ \int_x^y \left\{ \int_0^x \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} dz \right\} du =$$

$$= \int_0^x \frac{\lambda z}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda z)^i}{i!} e^{-\lambda z} \right\} dz + \int_0^x \frac{\lambda u}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} du +$$

$$+ \int_x^y \left\{ x \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} \right\} du =$$

$$= 2 \int_0^x \frac{\lambda z}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda z)^i}{i!} e^{-\lambda z} \right\} dz + \int_x^y \left\{ x \frac{\lambda}{n(n-1)} \left\{ \sum_{i=0}^{n-2} \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} \right\} \right\} du =$$

$$= 2 \frac{1}{n(n-1)} \sum_{i=0}^{n-2} (i+1) \int_0^x \lambda \frac{(\lambda z)^{i+1}}{(i+1)!} e^{-\lambda z} dz + x \frac{\lambda}{n(n-1)} \sum_{i=0}^{n-2} \int_x^y \lambda \frac{(\lambda u)^i}{i!} e^{-\lambda u} du =$$

$$= 2 \frac{1}{n(n-1)} \sum_{i=0}^{n-2} (i+1) \left\{ 1 - \sum_{j=0}^{i+1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \right\} + x \frac{\lambda}{n(n-1)} \sum_{i=0}^{n-2} \left\{ \sum_{j=0}^{i} \frac{(\lambda x)^j}{j!} e^{-\lambda x} - \sum_{j=0}^{i} \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right\} =$$

$$= 1 - 2 \frac{1}{n(n-1)} \sum_{i=0}^{n-2} (i+1) \left\{ \sum_{j=0}^{i+1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \right\} + x \frac{\lambda}{n(n-1)} \sum_{i=0}^{n-2} \left\{ \sum_{j=0}^{i} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \right\} -$$

$$- x \frac{\lambda}{n(n-1)} \sum_{i=0}^{n-2} \left\{ \sum_{j=0}^{i} \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right\} = 1 - 2 \frac{1}{n(n-1)} \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \left\{ \sum_{i=j-1}^{n-2} (i+1) \right\} +$$

$$+ x \frac{\lambda}{n(n-1)} \sum_{j=0}^{n-2} \left\{ \sum_{i=j}^{n-2} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \right\} - x \frac{\lambda}{n(n-1)} \sum_{j=0}^{n-2} \left\{ \sum_{i=j}^{n-2} \frac{(\lambda y)^j}{j!} e^{-\lambda y} \right\} =$$

$$= 1 - 2 \frac{1}{n(n-1)} \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} e^{-\lambda x} \left\{ \frac{1}{2} (n-j)(n+j-1) \right\} +$$

$$+ x \frac{\lambda}{n(n-1)} \sum_{j=0}^{n-2} \frac{(\lambda x)^j}{j!} e^{-\lambda x} (n-j-1) - x \frac{\lambda}{n(n-1)} \sum_{j=0}^{n-2} \frac{(\lambda y)^j}{j!} e^{-\lambda y} (n-j-1).$$

The following simplification gives:

$$G(x,y) = 1 - 2\frac{1}{n(n-1)}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}\left\{\frac{1}{2}(n-j)\big[(n-1)+j\big]\right\} +$$

$$+ x\frac{\lambda}{n(n-1)}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x}\big[(n-1)-j\big] - x\frac{\lambda}{n(n-1)}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y}\big[(n-1)-j\big] =$$

$$= 1 - \frac{1}{n}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}(n-j) - \frac{1}{n(n-1)}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}\big[(n-1)-(j-1)\big]j +$$

$$+ x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x} - x\frac{\lambda}{n(n-1)}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x}j - x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y} + x\frac{\lambda}{n(n-1)}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y}j =$$

$$= 1 - \sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x} + \frac{1}{n}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}j - \frac{1}{n}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}je^{-\lambda x} + \frac{1}{n(n-1)}\sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x}(j-1)j +$$

$$+ x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x} - x\frac{\lambda}{n(n-1)}\sum_{j=1}^{n-2}\frac{(\lambda x)^j}{(j-1)!}e^{-\lambda x} - x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y} + x\frac{\lambda}{n(n-1)}\sum_{j=1}^{n-2}\frac{(\lambda y)^j}{(j-1)!}e^{-\lambda y} =$$

$$= 1 - \sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x} + \frac{1}{n(n-1)}\sum_{j=2}^{n-1}\frac{(\lambda x)^j}{(j-2)!}e^{-\lambda x} + x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x} -$$

$$- x\frac{\lambda}{n(n-1)}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{(j-1)!}e^{-\lambda x} - x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y} + xy\frac{\lambda^2}{n(n-1)}\sum_{j=0}^{n-3}\frac{(\lambda y)^j}{j!}e^{-\lambda y} =$$

$$= 1 - \sum_{j=0}^{n-1}\frac{(\lambda x)^j}{j!}e^{-\lambda x} + x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda x)^j}{j!}e^{-\lambda x} - x\frac{\lambda}{n}\sum_{j=0}^{n-2}\frac{(\lambda y)^j}{j!}e^{-\lambda y} + xy\frac{\lambda^2}{n(n-1)}\sum_{j=0}^{n-3}\frac{(\lambda y)^j}{j!}e^{-\lambda y}.$$

The proof is ended.

References

1. Abegaz, F., Naik-Nimbalkar, U.V. Modeling statistical dependence of Markov chains via copula models, *Journal of statistical planning and inference*, Vol. **138**, 2008, pp. 1131-1146.
2. Andronov, A. On a simple method of multivariate distribution estimation defined by copulas. In: *Modelling in Business, Industry and Transport: Proceeding, International Conference, Riga, Latvia, May, 2008, Transport and Telecommunication Institute.* Riga: TTI, 2008, pp.233-241 .
3. Gertsbakh, I. *Reliability Theory: with Applications to Preventive Maintenance*. Berlin: Springer, 2000.
4. Embrechts, P. *Some Selected Papers*. – http://www.math.ethz.ch/~baltes/ftp/papers.html
5. Limnios, N., Nikulin, M. *Recent Advances in Reliability Theory: Methodology, Practice, and Inference.* Boston: Birkhauser, 2000. 514 p.
6. Paramonov, Yu., Andersons, J. A new model family for the strength distribution of fibers in relation to their length, *Mechanics of Composite Material*, Vol. 42(2), 2006, pp. 179-192.
7 Paramonov, Yu., Andersons, J. A family of weakest link models for fibers strength distribution, *Composite: Part A,* Vol. 38, 2007, pp. 1227-1233.
8 Ross, S.M. *Stochastic Processes*. New York: Wiley, 1996.
9. Sklar, A. Fonctions de répartition à *n* dimensions et leurs marges, *Publ. Inst Statist. Univ. Paris,* Vol. 8, 1959, pp. 229-231.
10. *The 8th Tartu Conference on MULTIVARIATE STATISTICS / the 6th Conference on MULTIVARIATE DISTRIBUTIONS* – http://www.ms.ut.ee/tartu07/presentations