A mathematical model is a description of a system using mathematical concepts and language. The process of developing mathematical model is termed mathematical modelling. Mathematical models are used in logistics systems analysis. A model helps to explain a system, study the effects of different components, and make predictions about behaviour. Mathematical models can take forms such as dynamical systems, statistical methodologies, differential equations, queuing theory, mathematical programming and others. Mathematical models can be divided into two types: analytical and simulation models.

This paper will review the contributions of mathematics to logistics and illustrate how the use of new mathematical insights and procedures will ensure the potential for continued successes in logistics.

Keywords: mathematical models, logistics systems analysis

1. Introduction

Mathematical modelling gets more and more important for logistics systems analysis as systems get complex and there is a need for a tool that helps to understand the systems and to give the desired answers to questions in time as short as possible. There are literature sources that describe how mathematical models are applied for logistics systems analysis. This paper will review the contributions of mathematics to logistics and illustrate how the use of new mathematical insights and procedures will ensure the potential for continued successes in logistics. In the second section is viewed development of logistics over the years. The third section envisaged disciplines of mathematics that are used for modelling and analysing logistics systems. The forth section is devoted for logistic systems management.

2. Development of Logistics

The last forty years has undergone profound and far-reaching changes in the function of logistics from its heavy concentration on a company’s physical processes to a holistic process and customer-oriented management. That led to continual researches and contributions of the mathematical applications in logistics. An overview of the development of logistics over time and the related areas of optimization is visible on Figure 1 (Baumgarten and Walter, 2000).

Classical logistics in the 1970s was mainly concerned with the flow of materials and goods. The aim was to guarantee the availability of materials and goods in the production process. The 1980s were characterized by the development of logistics management. Logistics had a cross-section function with the aim of optimizing functional comprehensive sequences. Logistics management optimize logistics services by integral consideration of previously separately planned and headed functions. Functional integration and company comprehensive integration are the two development recorded in the 1990s, when transformation from a functional perspective towards a flow-oriented perspective occurred. Construction and optimization of process chains as functional integration and of value chains as company comprehensive integration were the focus of interest. Today, logistics goes beyond a single company’s borders and that is the worldwide integration of value chains. The horizon of logistics increased from efficient flow management of materials and goods up to comprehensive, customer-oriented optimization of total supply chains. Logistics is a strategic instrument for business management and a decisive factor in competition (Neubauer, 2011).

Globalization of logistics, innovative network structures and co-operations, rising demand for superior logistics services, growing demand for customized logistics solutions, logistics as the key for cost reduction, trend „outsourcing” persists, accumulated and intensified safety regulations, rising importance of intelligent information and communication technologies, increasing requirements for the logistics staff improved flexibility of labour, change of the economic structure from the logistical point of
view, sustainable management of logistics systems, integration of logistics in the business strategies, competitive and locational advantages of logistics are the factors for impact on logistics and the contribution to a sustainable development (Flotzinger et al., 2008). Global logistics entails trade-offs in facility location, distribution networks, the routing and scheduling of deliveries by different modes of travel (e.g., air, water, truck, rail), procurement, and the overall management of international supply chains (Bookbinder, 2012).

To understand the contributions of mathematics to logistics, it is helpful to look at definitions of logistics that have been offered throughout its continual development. Modern logistics comprises operative logistics, analytical logistics and management of performance networks (Gudehus, 2009). The task of operative logistics or the four rights of logistics states that logistics has to provide the right quantities of goods most efficiently at the right place in the right order within the right time (see Figure 2).
Tasks of analytical logistics are to design optimal supply networks and logistics systems, to develop strategies for planning, scheduling and operation and to organize efficient order and performance processes. Daganzo (1999) mentioned that tasks of analytical logistics are to develop and organize optimal processes, structures, systems and networks for the operative logistics. Logistic management has to plan, implement and operate performance networks and to schedule the orders, resources and inventories.

3. Disciplines of Mathematics used to Model and Analyse Logistics Systems

Mathematics very often supports the transfer of knowledge between different scientific disciplines and industrial applications that are otherwise unrelated. Mathematical models (see Figure 3) are used to describe observations of the real world, physical or social processes in the language of mathematics.

<table>
<thead>
<tr>
<th>System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experiment with the actual system</td>
</tr>
<tr>
<td>Experiment with a model of the system</td>
</tr>
<tr>
<td>Physical model</td>
</tr>
<tr>
<td>Mathematical model</td>
</tr>
<tr>
<td>Analytical solution</td>
</tr>
<tr>
<td>Simulation</td>
</tr>
</tbody>
</table>

*Figure 3. Ways to Study a System*

Once we have built a mathematical model, it must then be examined to see how it can be used to answer the questions of interest about the system it is supposed to represent. If the model is simple enough, it may be possible to work with its relationships and quantities to get an exact, analytical solution. Some analytical solutions can become extraordinarily complex, requiring vast computing resources. If an analytical solution to a mathematical model is available and is computationally efficient, it is usually desirable to study the model in this way rather than via a simulation. However, many systems are highly complex, so that valid mathematical models of them are themselves complex, precluding any possibility of an analytical solution. In this case, the model must be studied by means of simulation, i.e., numerically exercising the model for the inputs in question to see how they affect the output measures of performance (Law and Kelton, 1999).

The last forty years have been very important in logistics developments field, from its concentration on a company’s physical processes to a holistic process and customer oriented management instrument. That also meant a continual change of the mathematical challenges in logistics (Möhring and Schenk, 2010). The importance of mathematical methods for logistics systems analysis confirms separate sections of international conferences or even ongoing individual conferences of this theme.

Application areas of mathematics for logistics problems are numerous. Figure 4 shows a list of particular kinds of logistics problems for which mathematics has been found to be a useful and powerful tool.

Perumbeva (2005) have given a brief overview of mathematical problems in logistics. The *Queuing theory* is an important tool used to model many supply chain problems. It is used to study situations in which customers or orders placed by customers form a line and wait to be served by a service or manufacturing facility. Long lines result in high response times and dissatisfied customers.

The Queuing theory may be used to:
- determine capacity required by manufacturing and service facilities within a supply chain;
- determine work-in-process inventory in a manufacturing facility;
- determine the average number of customers waiting to be served in a service facility;
- obtain information about service levels and shortages in manufacturing and service facilities;
- determine lead times between stages i.e. the time taken by one stage in a supply chain to satisfy an order placed by the next stage.
### Areas and Activities of Logistics

<table>
<thead>
<tr>
<th>Set-up</th>
<th>Planning</th>
<th>Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>1. Entire network</td>
<td>2. Individual site</td>
<td>3. Transport system</td>
</tr>
<tr>
<td>4. Supply chain</td>
<td>5. Supplier</td>
<td>6. Production site</td>
</tr>
<tr>
<td>7. OEM</td>
<td>8. Warehouse or transition point</td>
<td>9. Customer</td>
</tr>
</tbody>
</table>

### Logistical Problems

- a. Product line analysis and classification of articles – 1A, 1B
- b. Demand prognosis – 9A, 9B
- c. Location problem – 1A, 1B
- d. Selection of transport routes – 1A, 1B
- e. Set-up of supply chains – 1A, 1B
- f. Route planning – 3B, 3C, 4B, 4C
- g. Layout planning – 6A, 7A
- h. Storage and replenishment strategies – 5C, 6C, 7C
- i. Supply strategies – 5C, 6C, 7C
- j. Storage organization and operation strategies – 8B, 8C
- k. Set-up and dimensioning of vehicle systems – 6A, 7A, 8A
- l. Set-up of consignment processes – 8A, 8B
- m. Formation of logistic units – 8B, 8C
- n. Marginal efficiency and congestion effects – 6A, 6B, 8A, 8B
- o. Order scheduling and production planning – 5B, 5C, 6B, 6C, 7B, 7C
- p. Sequence problems – 6B, 6C, 7B, 7C

### Areas of Mathematics

- Linear (integer) optimization – c, d, e, f, g, k, l, m, o, p
- Heuristic optimization methods – c, d, e, f, g, k, l, m, o, p
- Scheduling algorithms – l, p
- Special analytic models – a, b, h, i, l, m
- Graph theory – c, d, e, f, k
- Automata theory and Petri nets – o
- Mathematical statistics – a, b, i, k, n
- Queueing models – n
- Continuous simulation models – e, h, i
- Discrete event simulation models – g, h, i, k, l, n, o, p

*Figure 4.* Mathematics in Logistical Problems

Statistical methodologies are used to make predictions about the level of consumer demand and the extent of unpredictability of consumer demand. They form the basis of Statistical Process Control techniques used for the monitoring of quality levels. Data mining techniques based partly on statistical tools are used to look for patterns and relationships in a body of data. One important application of data mining is in the analysis of marketing data to determine consumer behaviour patterns.

Statistical methodologies can be applied for:

- monitoring quality levels and identifying quality problems;
- setting up programs for improvement in quality;
- analysing marketing data to determine consumer behaviour patterns;
- forecasting future sales:
  - obtain information about average demand for a product or service over a period of time;
  - obtain information about the relationships between demand levels for various products and services;
  - obtain information about short-term and long-term demand trends;
• obtain information about demand seasonality;
• identify the underlying factors that drive consumer demand and the relative magnitude and importance of each factor;
  o obtain information about the level of unpredictability in demand for a product or service to set safety stock levels.

Moore and Ray (1999) have presented several statistical techniques that can be used to investigate sensitivity and performance analysis of simulation models. Formal statistical methods such as analysis of variance-based methods and regression tree analysis are used to determine variables having substantive influence on the experimental results and to investigate the structure of the underlying relationship between inputs and outputs.

**Simulation** is used to study situations characterized by uncertainty. Simulation involves the creation of a model of a system based on specific assumptions about system behaviour and information about probability distributions associated with various variables. By running simulations, it may be possible to determine the best values of various system parameters, subject to the underlying assumptions. The main advantage of simulation is that it can be used to study extremely complex systems that cannot be easily modelled by using other mathematical tools. It must be emphasized that the recommended solutions are a function of the quality of the model and the underlying assumptions.

Simulation of logistics systems can be used to:
  o determine the location and capacity of manufacturing and service facilities within a supply chain;
  o compare expected system performance and total costs for various configurations of a supply chain;
  o determine lead times within a supply chain.

The increasing use of computer simulation methods for modelling complex manufacturing processes had led to the need for statistical methods that can be used to understand such systems. Simulation model of a manufacturing supply chain may contain many different process activities, such as receiving, processing, transporting, inspecting. Each of these activities may have many different variables whose values can affect the resulting process performance, including the inventory replenishment strategy, manufacturing orientation or capacity planning strategy. Outputs of interest may include, among others, measures of manufacturing cycle time, supplier lead time, number of lost customers and profits.

**Mathematical programming** includes linear programming, nonlinear programming, and integer programming. A mathematical programming problem consists generally of an objective (often the maximization of profits or the minimization of costs) and a set of constraints (for instance, a limited budget or capacity) faced by the decision-maker. Usually there are sets of variables that are under the control of the decision-maker. There are also parameters that are fixed, values that are not under the control of the decision-maker. Mathematical programming techniques make it possible for a decision-maker to determine the values of variables that result in the optimal solution. These mathematical tools are used to obtain solutions to specific problems in supply chains.

**Linear programming** tool is used for situations when a manager must allocate limited resources to maximize the profits, market share, or sales revenue or minimize the costs, defects and the objective (e.g. sales revenue) is proportional to the values of certain variables (e.g. the number of units sold). Such situations include the allocation of the following:
  o limited amount of manufacturing capacity to a set of products;
  o limited amount of raw material, components or subassemblies to the production of various products;
  o limited amount of shelf space at a retail facility to various products;
  o capital budget to various kinds of capacity acquisition and investment alternatives.

**Nonlinear programming** tool is used in situations when the objective does not vary linearly with the values of certain variables. This includes situations in which there exist economies or diseconomies of scale, such as:
  o capacity acquisition in the presence of significant economies of scale;
  o determine the appropriate level of overtime in situations in which quality and employee morale exhibit diseconomies of scale i.e. they are adversely affected at high levels of
overtime, and wages exhibit nonlinearities because overtime pay exceeds regular-time pay;
- determine the appropriate level of machine utilization in situations in which available machine time decreases and costs increase at high levels of utilization due to increased wear-and-tear and machine breakdowns;
- assign tasks to a set of workers in situations in which a significant learning curve effect exists for certain tasks.

Integer programming tool is used for situations in which a yes or no decision must be made, and situations in which fractions of an object serve no purpose. This includes:
- choosing an appropriate location for a manufacturing facility, warehouse or service facility;
- deciding whether to shut down a manufacturing facility, warehouse or service facility;
- developing production schedules and determining the sequence of production of various products at a manufacturing facility;
- developing workforce schedules and timetables at a manufacturing or service facility;
- making a choice between infrequent full truckload shipments and frequent less-than-truckload shipments;
- determining the optimal set of routes for a fleet of trucks and a given set of drop-off points;
- determining appropriate purchase quantities and an appropriate purchase schedule in situations in which suppliers offer price discounts for bulk purchases;
- determining the optimal number of machines in a manufacturing facility;
- determining the size and composition of a fleet of trucks consisting of trucks of various dimensions and capacities;
- decisions relating to the introduction of a new product or the phasing out of an old product;
- making a choice between performing an activity or business process in-house versus outsourcing the activity or business process.

To obtain a more strategic perspective researchers have often used the tools of game theory. Game theory may be used to model negotiation processes in supply chains and to develop insights into the balance of power in supply chains including:
- determining the optimal strategy to be adopted during negotiation of supply chain contracts;
- determining the appropriate relationship with a supplier or customer deciding whether vertical integration (taking over a supplier or a customer) is appropriate;
- making a choice between using a single source or multiple sources for a component or raw material;
- determining the appropriate level of competition within a firm;
- determining the impact of incentive systems and performance measurement systems on the level of coordination between different functional areas (e.g. between manufacturing and marketing) in a firm.

4. Logistic Systems Management

Management science has been known by a variety of other names, for example, operations research. Some people tend to identify the scientific approach to managerial problem solving under such other names as system analysis, cost-benefit analysis, and cost-effectiveness analysis. Operations research is focused on solving practical problems that can be described by mathematical model. Taha (2007) in his book presents the main aspects of the operations research theory: mathematical programming (linear and nonlinear, deterministic and stochastic), decision making theory and game theory, inventory management theory, queuing theory, simulation modelling. Domschke (2007) in his books looks at operations research problems in transport, traveling and tourism, as well as locations.
Logistics decisions may be divided or grouped in several dimensions based on various criteria. Langevin and Riopel (2005) presented a three-part decision hierarchy consisting of a Strategic Planning level, a Network level and an Operations level (see Figure 5).

![Figure 5. Logistics Decision Categories]

For many companies, the ability to efficiently match demand and supply is the key to their success. Failure could lead to loss of revenue, reduced service levels, impacted reputation, and decline in the company’s market share. Recent developments such as market competition intense, proliferation of product, and the increase in the number of products with a short life cycle have created an environment where customer demand is volatile and unpredictable. In such an environment, traditional operations strategies such as building inventory, investing in capacity buffers, or increasing committed response time to consumers do not offer a competitive advantage. Therefore, many companies are looking for effective strategies to respond to market changes without significantly increasing cost, inventory, or response time. This has motivated a continuous evolution of the management of logistics systems (Levi et al., 2014).

Supply Chain management is the management of the flow of goods or services from materials stage to the end user, is a complex process because of the level of uncertainty at each stage of the supply chain. Computer simulation, because it can be applied to operational problems that are too difficult to model and solve analytically, is an especially effective tool to help analyse supply chain logistical issues. The most effective supply chain management systems are designed to deliver high-quality products and services promptly and reliably at the least cost. To accomplish this goal, all the supply chain processes must be effectively coordinated. Currently, tools for understanding uncertainty are limited to traditional mathematical formulas that do not account for variability. Simulation is one of the best means for analysing supply chains because of its capability for handling variability. Companies can use simulation to see how effective and costly an inventory system would be in their own environment without having to implement the system physically. Optimal acquisition cost, capitalization cost, average labour hours, average flow time, and the number of shells assembled are one of the outputs that could be found using simulation optimization tool (Schunk and Plott, 2000). The “hottest” topic in discrete-event simulation today is simulation-based optimization. Law and McComas (2000) look closer at optimisation of manufacturing problem using two commercial packages – Optquest and WITNESS. Based on the availability of faster PCs and improved heuristic optimization techniques (genetic algorithms, simulated annealing, tabu search, etc.) most discrete-event simulation software vendors have integrated optimization packages into their simulation software and nowadays it is not a problem to do simulation based optimization. The goal of an optimization package is to orchestrate the simulation of a sequence of system configurations (each configuration corresponds to particular settings of the decision variables) so that a system configuration is eventually obtained that provides an optimal or near optimal solution.

Daganzo (2005) described how to plan and design efficient logistics systems considering simultaneously all integral aspects of their operations, and how to evaluate economically existing or proposed systems. The methodologies are useful when decisions have to be made with incomplete or uncertain information, e.g. when evaluating a business plan, or designing a system for a long time horizon. The author has envisaged optimization methods for logistics systems with one origin and one destination, many origins and one destination, many destinations and one origin, and many origins and many destinations for systems with and without transshipments.
The logistics management issues that span a large spectrum of decisions are network configuration, production planning, inventory control and pricing optimization, procurement strategies and supply contracts, process flexibility, integration of production, inventory and transportation decisions, vehicle fleet management, truck routing and packing problems (Simchi-Levi et al., 2014). Due to the fact that as the models become more complex and integrate more and more issues that arise in practice, their analysis becomes more difficult. Some problems require the use of methods from game theory in order to understand how different supply chain partners respond to various challenges. Other problems have at their core difficult combinatorial problems in the class called NP-hard problems. This implies that it is very unlikely that one can construct an algorithm that will always find the optimal solution, or the best possible decision, in computational time that is polynomial in the size of the problem. Therefore, in many cases, an algorithm that consistently provides the optimal solution is not considered a reachable goal and hence heuristic or approximation methods are employed. For assessing and quantifying a heuristic’s effectiveness the following methods may be employed: empirical comparisons, worst case analysis, average case analysis.

Supply chain is a complex system, as well as a system of systems, because it embeds other systems representing product, process, and organizational structures of an enterprise. Three system facets applied to the supply chain are depicted in Figure 6 (Chandra and Grabis, 2007).

Figure 6. General System Facets Applied to a Supply Chain

From the system management perspective, a supply chain is viewed as an organizational system, because a supply chain has managerial issues that can be classified into three levels – strategic, tactical, and operational.

<table>
<thead>
<tr>
<th>Decision-Making Level</th>
<th>Timeline</th>
<th>Type of Decision Made</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strategic</td>
<td>3 to 10 years</td>
<td>Investment on plants and capacities.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Introduction of new products.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Creation of a logistics network.</td>
</tr>
<tr>
<td>Tactical</td>
<td>3 months to 2 years</td>
<td>Inventory policies to use.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Procurement policies to be implemented.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transportation strategies to be adopted.</td>
</tr>
<tr>
<td>Operational</td>
<td>Day to Day</td>
<td>Scheduling of resources.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Routing of raw materials and finished products.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Solicitation of bids and quotations.</td>
</tr>
</tbody>
</table>

Logistics management is that part of the business that plans, implements, and controls the efficient forward and reverse flow and storage of goods, services and related information between the point of origin and the point of consumption in order to meet customers’ requirements. Supply networks consist of delivery sites and receiving sites that are connected to one another through supply chains and depending on the process in question are called supply chains, acquisition chains, transport chains, cargo chains, shipping chains or disposal chains, or logistical chains in general (Gudehus, 2009). In logistics systems items are produced at one or more factories, shipped to warehouses and distribution centres for intermediate storage, and then shipped to retailers or customers. Consequently, to reduce cost and improve service levels, logistics strategies must take into account the interactions of these various levels in this logistics network, also referred to the supply chain. This network consists of suppliers, manufacturing centers, warehouses, distribution centers, and retailer outlets, as well as raw materials, work-in-process inventory, and finished products that flow between the facilities, see Figure 7 (Levi et al., 2014).
Firstly, logistics management takes into consideration every facility that has an impact on cost and plays a role in making the product conform to customer requirement. Secondly, the objective of logistics management is to be efficient and cost-effective across the entire system; from transportation and distribution to inventories of raw material, work-in-process, and finished goods, is to be minimized. So the emphasis is not on simply minimizing transportation cost or reduction inventories but, rather, on taking a systems approach to logistics management.

5. Conclusions

The importance of mathematical methods for logistics systems analysis nowadays confirms separate sections of international conferences or even ongoing individual conferences of this theme. The International Symposium on Mathematics of Logistics took place at Tokyo University of Marine Science and Technology (2011), where scientists and practitioners had opportunities to attend lectures on theory and practices of mathematical methods of logistics. Also the Winter Simulation Conference attracts many researchers in this field every year.

There are institutions that providing services for logistics performance improvements, for example, in Australia there is the Centre for Industrial Modelling and Optimisation (CIMO), which specialises in industry-focused research and training, it offers consulting services in applying optimisation, operations research and statistics to industries such as transport. There are many challenging optimisation problems in the design and operation of transport and logistics networks. The activities of the research are applied to vehicle routing and scheduling for long and short haul operations; optimal fleet sizing, composition, maintenance and replacement; optimal warehouse operations; supply chain management; and a wide range of other logistics issues.

References