Dynamic Traffic Assignment models and the problem of their calibration

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Outline

• Introduction / Motivation

• Calibration framework

  – Formulation: (i) general, (ii) off-line, (iii) on-line
  – Solution approaches

• Case studies

  – DTA, off-line, Los Angeles, CA and Lower Westchester County, NY
  – DTA, on-line, Southampton, UK
  – Microscopic traffic simulator, Lower Westchester County, NY

• Directions for further research
Motivation

Data

Simulation model

Calibration
"For 50% of road projects, the difference between actual and forecasted traffic is more than 20%; for 25% of road projects, the difference is larger than 40%.” (Source: Flyvbjerg, B., M. K. Skamris Holm, and S. L. Buhl (2006). Inaccuracy in traffic forecasts. Transport Reviews, Vol. 26, No. 1, 1-24, January 2006.)
Motivation

Source: Dilbert.com
On-line DTA framework

- Surveillance information
- A-priori parameter values
- Network representation
  - Historical data

- Demand simulator
- Supply simulator

State estimation and model calibration

- Demand simulator
- Supply simulator

State prediction

Network performance
Motivation (seriously now)

• Models/components
  – Demand–side
    * OD flows
    * Behavioral models (e.g. route choice)
  – Supply–side
    * Speed–density relationship (e.g. \( u = u_f \left[ 1 - \left( \frac{\max(0, K - K_{\text{min}})}{K_{\text{jam}}} \right)^\beta \right]^\alpha \))
      - where \( u \) denotes the speed, \( u_f \) is the free flow speed, \( K \) is the density, \( K_{\text{min}} \) is the minimum density, \( K_{\text{jam}} \) is the jam density and \( \alpha \) and \( \beta \) are model parameters.
    * Segment capacities

• Often, state estimation and model calibration approaches include subset of these models
Research objectives

• Develop an integrated approach for calibration that
  – Jointly estimates and predicts demand and supply parameters
  – Is general and flexible
    * Not tied to a particular simulation system
    * Applicable to any calibration parameters
    * Can exploit any available information
  – Is computationally feasible

• Demonstrate the feasibility of the approach
**Inputs and outputs**

- **A priori parameter values**
- **"Static" data (network etc.)**
- **Off-line calibration**
  - **Surveillance data (archived)**
  - **Off-line calibrated parameters**
- **"Static" data (network etc.)**
- **On-line calibration**
  - **Surveillance data (latest)**
  - **On-line calibrated parameters**
Integrated calibration framework

Calibration inputs:
DATA

- Archived Data (Multiple Days)
  - Sensor data
  - Special events log
  - Incident log
  - Weather log

- Off-line calibrated parameters

Calibration outputs:
MODEL INPUTS and PARAMETERS

- OD flows
- Capacities
- Traffic dynamics parameters
- Route choice parameters
- Travel times
- Error covariances
- OD prediction parameters

Real-Time Data
- Sensor data
- Special events
- Incidents
- Weather

OFF-LINE CALIBRATION

OFF-LINE/PLANNING APPLICATIONS

ON-LINE CALIBRATION

REAL-TIME/OPERATIONAL APPLICATIONS

Historical Database

Updating
Motivation

Source: Dilbert.com
Integrated calibration framework

A priori estimates of model inputs and parameters, if available, contain valuable structural information about the calibration variables and represent direct measurements:

\[ x^a = x + v \]
\[ \beta^a = \beta + w \]

\( x \) and \( \beta \) are vectors of the OD flows and model parameters to be calibrated respectively; \( x^a \) and \( \beta^a \) are the corresponding a priori values, \( v \) and \( w \) are error vectors.
Integrated calibration framework

Formulation as an optimization problem

$$(\beta^*, x^*) = \arg \min_{\beta, x} z(M, M, x, x^a, \beta, \beta^a)$$

In the above expression, $x_h$ is the vector of OD flows departing in interval $h$; $x^a_h$ are initial OD flow estimates; $\beta_h$ is a vector of route choice and supply model parameters; $\beta^a_h$ are the corresponding initial values; $x = \{x_h | h \in \tau\}$; $x^a = \{x^a_h | h \in \tau\}$; $\beta = \{\beta_h | h \in \tau\}$; $\beta^a = \{\beta^a_h | h \in \tau\}$. 
Off–line vs. on–line calibration

- Off–line calibration is a global optimization step. On-line calibration, on the other hand, attempts to dynamically adjust the off-line calibrated parameter values.

- Large archived databases of prior observations may be available for the off-line calibration while on-line calibration uses the most recent time period.

- While off-line calibration has no explicit computational requirements, on-line calibration has to be performed in real time.

- Off-line calibration must involve the estimation of a complete set of simulation model inputs and parameters, while on-line calibration may focus on a subset of parameters.
**Off-line calibration formulation**

Formulated as an optimization problem

$$(\beta^*, x^*) = \arg \min_{\beta, x} \sum_{h=1}^{H} \left[ z_1(M_h, M_h, \beta, \ldots, \beta_h; G_1, \ldots, G_h) + z_2(x_h, x_h^a) + z_3(\beta_h, \beta_h^a) \right]$$

subject to:

$${M}_h = f(x_1, \ldots, x_h, \beta_1, \ldots, \beta_h; G_1, \ldots, G_h)$$

$$l_h^x \leq x_h \leq u_h^x$$

$$l_h^\beta \leq \beta_h \leq u_h^\beta$$
Off-line calibration — Solution approaches

• Path search
  – Stochastic Approximation
    * Finite Difference Stochastic Approximation
    * Simultaneous Perturbation Stochastic Approximation
  – Response Surface Methodology
    * Involves local polynomial approximation of the surface at each iteration, extensions, such as SNOBFIT

• Pattern search, e.g. Hooke and Jeeves, Nelder and Mead’s Downhill Simplex

• Random search methods, including metaheuristics such as
  – Genetic Algorithms
  – Simulated Annealing
Off-line calibration — Solution approaches: SPSA

• Simultaneous Perturbation Stochastic Approximation
  – Iteratively updates \( \theta \)

\[
\theta^{i+1} = \theta^i - a^i \hat{g}^i(\theta^i)
\]

– Gradient approximation

\[
\hat{g}_k^i = \frac{z(\theta^i + c^i \Delta_i) - z(\theta^i - c^i \Delta_i)}{2c^i \Delta_{ik}}
\]

where \( \theta = [x; \beta] \)
On-line calibration formulation

- State-space model
  - State vector
  - Measurement equations
  - Transition equations

- Idea of deviations
State vector

• Deviations $\Delta \pi_h$ of model inputs and parameters $\pi_h$ from available estimates

$$\Delta \pi_h = \pi_h - \pi^H_h$$

• State vector includes
  
  – OD flows $x_{ijh}$: number of vehicles departing from origin $i$ towards destination $j$ during time interval $h$
  
  – Model parameters, e.g.
    
    * route choice model parameters
    * speed–density relationship parameters
    * segment capacities
Measurement equations (I)

• Direct measurement equation

\[ \pi^a_h = \pi_h + v_h \]

• In deviations’ form (subtracting \( \pi^H_h \) from both sides)

\[ \pi^a_h - \pi^H_h = \pi_h - \pi^H_h + v_h \Rightarrow \]

\[ \Delta \pi^a_h = \Delta \pi_h + v_h \]

where

– \( a \) denotes \textit{a priori} values, \( H \) indicates historical estimates, \( v_h \) is a random error vector.
Measurement equations (II)

• Indirect measurement equation

\[ M_h = S(\pi_h) + \nu_h \]

• In deviations’ form (subtracting \( M_h^H \) from both sides)

\[
M_h - M_h^H = S(\pi_h) - M_h^H + \nu_h \Rightarrow \\
\Delta M_h = S(\pi_h + \Delta \pi_h) - M_h^H + \nu_h
\]

where

– \( M_h \) is a vector of measurements for interval \( h \), and \( \nu_h \) is a random error vector.

– \( S \) is the simulator function mapping the state to the measurements (no analytic expression).
Transition equations

\[ \pi_{h+1} = \sum_{q=h-p}^{h} F_{q}^{h+1} \cdot \pi_q + \eta_h \]

- In deviations’ form (subtracting \( \pi^H \) for appropriate interval from both sides)

\[ \pi_{h+1} - \pi_{h+1}^{H} = \sum_{q=h-p}^{h} F_{q}^{h+1} (\pi_q - \pi_q^{H}) + \eta_h \Rightarrow \]

\[ \Delta \pi_{h+1} = \sum_{q=h-p}^{h} F_{q}^{h+1} \cdot \Delta \pi_q + \eta_h \]

where

- \( F_{q}^{h+1} \) is a transition matrix of effects of \( \Delta \pi_q \) on \( \Delta \pi_{h+1} \)
- \( p \) is the degree of the autoregressive process, and
- \( \eta_h \) is a random error vector
The model at a glance

\[ \Delta \pi_h^a = \Delta \pi_h + v_h \]
\[ \Delta M_h = S(\pi_h^H + \Delta \pi_h) - M_h^H + v_h \]
\[ \Delta \pi_{h+1} = \sum_{q=h-p}^{h} F_q^{h+1} \cdot \Delta \pi_q + \eta_h \]

(Or, equivalently, as an optimization formulation:)

\[ \pi^*_h = \text{argmin}[N_1(e_h^a) + N_2(e_h^{obs}) + N_3(e_h^{auto})] \]
\[ = \text{argmin}[N_1(\pi_h - \pi_h^a) + N_2(M_h^{sim} - M_h^{obs}) + N_3(\pi_h - \sum_{q=h-p-1}^{h-1} F_q^{h+1} \pi_q)] \]
Solution approaches

• Linear models
  – Kalman Filter (KF)

• Non–linear models
  – Extended Kalman Filter (EKF)
    * Need numerical derivatives if no analytical expression
  – Limiting Extended Kalman Filter (LimEKF)
    * Use off-line pre-computed Kalman gain
    * Tomorrow: interesting results using different ways to select/compute
      the Kalman gain
  – Unscented Kalman Filter (UKF)
    * Same computational complexity as EKF
  – SP-EKF: Extended Kalman Filter using SPSA for numerical derivative
Kalman Filtering principles

• Prediction–correction approach

1. **Predict** *a priori* state for interval $h$ (using information up to $h - 1$)

2. Compute *Kalman gain* matrix

3. Combine Kalman gain and surveillance from interval $h$ to **correct** state prediction for interval $h$
Extended Kalman Filter

- Extension of Kalman Filter for non-linear models
  - First order Taylor expansion approximation

- Can use numerical derivatives, if no analytical relationship exists (for measurement equations)
  - Large number of function evaluations is required
  - e.g. $2n$ evaluations if central differences are used ($n$ is state dimension)

- Computation of Kalman gain most “expensive” operation
Limiting Extended Kalman Filter

- Use a pre-computed Kalman gain
  - E.g.: (weighted) average of Kalman gains (computed off-line)
  - No need to compute the Kalman gain on-line
  - Tomorrow: interesting results using different ways to select/compute the Kalman gain

- Only a single function evaluation is required on-line

- Can run EKF off-line and periodically re-compute Kalman gain
Case studies — Objectives

• Demonstrate feasibility of the calibration approach

• Verify importance of calibration
  – Impact of joint estimation and prediction of demand and supply parameters

• Test candidate algorithms based on several criteria
  – Performance/Fit (estimation and prediction)
  – Computational properties
  – Robustness
Case studies

• DTA, off-line calibration
  – (Downtown) Los Angeles, CA
  – Lower Westchester County, NY (with synthetic AVI data)

• DTA, on-line calibration
  – Southampton, UK

• Microscopic traffic simulator
  – Lower Westchester County, NY
The DynaMIT–R system

• State-of-the-art, simulation-based DTA system
  – Transportation network and supply characteristics
  – OD demand and travel behavior

• Operates in rolling horizon for:
  – Real-time estimation of network performance
  – Short-term prediction of future network conditions
  – Generation of traffic information

• Inputs
  – A-priori OD flows
  – Route choice parameters
  – Traffic dynamics models’ parameters
  – Segment capacities
DynaMIT–R GUI

10.4 Dynamic Traffic Assignment

DynaMIT combines the models presented in the previous section to estimate and predict the state of the traffic network and generate and disseminate traffic information.

Figure 10.7 outlines DynaMIT’s state estimation process, which reproduces a full description of the current network conditions to match the available real-time data as closely as possible and estimates the key model parameters that will be used for state prediction and information generation. To perform state estimation, DynaMIT uses historical OD matrices, the travel time information provided to the travelers, and real-time traffic counts from the surveillance systems. Disaggregate behavioral models use the traffic information to update the departure time and route choice decisions of the travelers, which are then aggregated into updated OD matrices. The online calibration model in the state estimation uses the updated demand information and the latest available surveillance data to estimate OD flows and key model parameters (such as segment capacities). This information is loaded into the supply simulator, where the en route demand models are also used. The estimated network conditions and key estimated parameter values are produced and used as input for the prediction-based information generation.

The prediction-based information generation module of DynaMIT is shown in Fig. 10.8. While no measurements of the future traffic state exist, the objective is to...
Off–line calibration: Los Angeles
Off–line calibration: Los Angeles

- 243 nodes / 606 links

- Archived freeway and arterial flow and speed data from 203 loop detectors

- 3:00 – 9:00 a.m. (divided into 15-min intervals)

- Fairly congested network with a large number of route choices for each of the 1129 OD pairs.

- Time- dependent OD flows were calibrated for each interval, together with route choice model parameters, segment capacities, and speed–density function parameters.
Off-line calibration (LA) — Case study results

The performance of the calibration methodology was assessed using the normalized root mean square error (RMSN) of the counts and speeds:

\[
\text{RMSN} = \frac{\sqrt{N \sum \left( y - \hat{y} \right)^2}}{\sum y}
\]

where \( N \) is the number of measurements, \( y \) denotes the observed, and \( \hat{y} \) the estimated traffic parameters, e.g. counts or speeds.

Table 10.2 Fit to counts and speeds, RMSN (off-line calibration)

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Fit to counts</th>
<th></th>
<th>Fit to speeds</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Freeway</td>
<td>Arterial</td>
<td>Freeway</td>
<td>Arterial</td>
</tr>
<tr>
<td>Reference</td>
<td>0.218</td>
<td>0.239</td>
<td>0.181</td>
<td>0.203</td>
</tr>
<tr>
<td>D</td>
<td>0.114</td>
<td>0.143</td>
<td>0.118</td>
<td>0.125</td>
</tr>
<tr>
<td>SD</td>
<td>0.090</td>
<td>0.113</td>
<td>0.088</td>
<td>0.093</td>
</tr>
</tbody>
</table>
COST/Multitude Workshop within the framework of RelStat2012 - Riga, Latvia
DTA, Off-line calibration – LWC

- 1767 links (split into 2564 segments) / 825 nodes / 482 origin-destination pairs / 55 point data locations / 10 AVI locations
- 6:00 am to 8:00 am divided into eight 15-minute intervals
- Total number of parameters for the case of joint demand-supply calibration is 3856 OD flows plus 2564 segment capacities plus 50 speed-density parameters equals 6470 parameters.

<table>
<thead>
<tr>
<th>Table 10.4</th>
<th>Calibration and validation results for Lower Westchester County network</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Demand-only calibration</td>
</tr>
<tr>
<td>RMSN</td>
<td>Link counts (%)</td>
</tr>
<tr>
<td>Counts</td>
<td>20.09</td>
</tr>
<tr>
<td>Travel Times</td>
<td>22.23</td>
</tr>
</tbody>
</table>
DTA, On–line calibration – Southampton, UK
Southampton, UK — Network
Southampton, U.K. (35km portion of freeway M27)

- 45 segments (three types)
- 10 sensors
- 20 OD pairs
- PM data (peak direction)
Traffic flow characteristics

Sensor counts (all days)

Counts (veh/hr/lane)

15:00 16:00 17:00 18:00 19:00

Sensor counts (all days)

Speeds/Densities (dry days)

Density (veh/km/lane)

15:00 16:00 17:00 18:00 19:00

Speeds/Densities (wet days)

Density (veh/km/lane)

15:00 16:00 17:00 18:00 19:00
On–line calibrated speed–density relationships
Mainline segments – EKF

top: dry weather, bottom: wet weather

Speed (kph)

Density (veh/km/lane)
**On–line calibration results**

**Table 10.3** Fit to counts and speeds

<table>
<thead>
<tr>
<th></th>
<th>Estimated</th>
<th>One-step predicted</th>
<th>Two-step predicted</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Speeds</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Online (EKF)</td>
<td>0.1106</td>
<td>0.1209</td>
<td>0.1303</td>
</tr>
<tr>
<td>Online (LimEKF)</td>
<td>0.1120</td>
<td>0.1249</td>
<td>0.1347</td>
</tr>
<tr>
<td><strong>Counts</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Online (EKF)</td>
<td>0.1039</td>
<td>0.1318</td>
<td>0.1550</td>
</tr>
<tr>
<td>Online (LimEKF)</td>
<td>0.1091</td>
<td>0.1321</td>
<td>0.1702</td>
</tr>
</tbody>
</table>

(Note: LimEKF robustness)
On–line calibration — Case study results
LimEKF (top: speeds, bottom: counts)

![Graph showing results](image)

**RMSN**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>0.25</td>
<td>0.15</td>
<td>0.10</td>
<td>0.08</td>
<td>0.07</td>
<td>0.06</td>
</tr>
<tr>
<td>LimEKF</td>
<td>0.20</td>
<td>0.12</td>
<td>0.09</td>
<td>0.07</td>
<td>0.06</td>
<td>0.05</td>
</tr>
</tbody>
</table>

COST/Multitude Workshop within the framework of RelStat2012 - Riga, Latvia
Computational performance

• EKF
  – $n + 1$ or $2n$ function evaluations (essentially forward or central numerical derivative, respectively)

• Limiting Kalman Filter
  – $O(1)$ computational performance
  – Single function evaluation required (independent of state dimension)
Off-line calibration — Microscopic traffic simulator

• Difference between various levels of simulation (micro-meso-macro)
  – Number of parameters / problem dimension
  – Stochasticity

• Demonstrate that the approach is flexible/robust
  – Framework
  – Solution approaches

• Case study setup
  – MITSIMLab (open source version available at: http://web.mit.edu/its/MITSIMLabOSnew.html)
  – Lower Westchester County network
  – Solution approach: SPSA
# Off-line calibration — Microscopic Traffic Simulator

<table>
<thead>
<tr>
<th>Period</th>
<th>Before</th>
<th>After</th>
<th>% change</th>
<th>Before</th>
<th>After</th>
<th>% change</th>
<th>Before</th>
<th>After</th>
<th>% change*</th>
</tr>
</thead>
<tbody>
<tr>
<td>6:00 – 6:15</td>
<td>0.537</td>
<td>0.179</td>
<td>67%</td>
<td>0.535</td>
<td>0.216</td>
<td>60%</td>
<td>-0.474</td>
<td>0.004</td>
<td>99%</td>
</tr>
<tr>
<td>6:15 – 6:30</td>
<td>0.219</td>
<td>0.158</td>
<td>28%</td>
<td>0.224</td>
<td>0.172</td>
<td>23%</td>
<td>-0.128</td>
<td>-0.006</td>
<td>95%</td>
</tr>
<tr>
<td>6:30 – 6:45</td>
<td>0.208</td>
<td>0.155</td>
<td>26%</td>
<td>0.190</td>
<td>0.148</td>
<td>22%</td>
<td>-0.137</td>
<td>-0.044</td>
<td>68%</td>
</tr>
<tr>
<td>6:45 – 7:00</td>
<td>0.263</td>
<td>0.198</td>
<td>25%</td>
<td>0.253</td>
<td>0.181</td>
<td>29%</td>
<td>-0.171</td>
<td>-0.052</td>
<td>70%</td>
</tr>
<tr>
<td>7:00 – 7:15</td>
<td>0.405</td>
<td>0.248</td>
<td>39%</td>
<td>0.381</td>
<td>0.225</td>
<td>41%</td>
<td>-0.271</td>
<td>-0.121</td>
<td>55%</td>
</tr>
<tr>
<td>7:15 – 7:30</td>
<td>0.528</td>
<td>0.245</td>
<td>54%</td>
<td>0.443</td>
<td>0.237</td>
<td>47%</td>
<td>-0.335</td>
<td>-0.118</td>
<td>65%</td>
</tr>
<tr>
<td>7:30 – 7:45</td>
<td>0.553</td>
<td>0.236</td>
<td>57%</td>
<td>0.454</td>
<td>0.217</td>
<td>52%</td>
<td>-0.353</td>
<td>-0.120</td>
<td>66%</td>
</tr>
<tr>
<td>7:45 – 8:00</td>
<td>0.546</td>
<td>0.160</td>
<td>71%</td>
<td>0.453</td>
<td>0.157</td>
<td>65%</td>
<td>-0.337</td>
<td>-0.062</td>
<td>82%</td>
</tr>
<tr>
<td>8:00 – 8:15</td>
<td>0.656</td>
<td>0.236</td>
<td>64%</td>
<td>0.638</td>
<td>0.230</td>
<td>64%</td>
<td>-0.614</td>
<td>-0.070</td>
<td>89%</td>
</tr>
<tr>
<td>8:15 – 8:30</td>
<td>0.435</td>
<td>0.249</td>
<td>43%</td>
<td>0.390</td>
<td>0.242</td>
<td>38%</td>
<td>-0.262</td>
<td>-0.065</td>
<td>75%</td>
</tr>
<tr>
<td>8:30 – 8:45</td>
<td>0.427</td>
<td>0.285</td>
<td>33%</td>
<td>0.409</td>
<td>0.294</td>
<td>28%</td>
<td>-0.199</td>
<td>-0.011</td>
<td>94%</td>
</tr>
<tr>
<td>8:45 – 9:00</td>
<td>0.420</td>
<td>0.268</td>
<td>36%</td>
<td>0.403</td>
<td>0.281</td>
<td>30%</td>
<td>-0.192</td>
<td>0.030</td>
<td>84%</td>
</tr>
<tr>
<td>6:00 – 9:00</td>
<td>0.476</td>
<td>0.231</td>
<td>51%</td>
<td>0.416</td>
<td>0.221</td>
<td>47%</td>
<td>-0.289</td>
<td>-0.053</td>
<td>82%</td>
</tr>
</tbody>
</table>

* in absolute values
Primary benefits of the approach

• Non-linear dependencies between traffic data and calibration variables are accurately captured
  – Consequently, linear approximations such as the assignment matrix mapping are not required.

• All demand and supply parameters ($x_h$ and $\beta_h$) may be estimated simultaneously and efficiently

• Any general traffic data can be included in $M_h$

• Capacity reduction factors can be estimated for known incidents in the observed data, to accurately incorporate supply impacts on calibration

• Supply parameters for links without sensors can be updated independently, using the parameters from the closest links only as starting values.
Directions for further research

• Further experimental analysis
  – Scalability
  – Robustness
  – Sensitivity

• Selection of parameters (primarily for on–line calibration)
  – Dynamic (?)

• Impact of grouping segments (for speed–density relationship estimation)

• Algorithms
  – e.g. particle filters (generalization of UKF)
  – parallel versions of algorithms
"Controversial" point/question

What are the boundaries between off-line and on-line calibration?

- Do we need both?
- Given a reasonable warm-up period, do we need off-line calibration?
- Unless there are disruptions, is on-line calibration a waste of time?
Indicative references


Thank you

Questions?

Contact information:
Constantinos Antoniou (antoniou@central.ntua.gr)

Complete list of publications can be found at:
http://users.ntua.gr/antoniou/publications.html
Backup slides
Kalman Filter

Initialization

\[ X_{0|0} = X_0 \]
\[ P_{0|0} = P_0 \]

**for** \( h = 1 \) to \( N \) **do**

Time update

\[ X_{h|h-1} = F_{h-1}X_{h-1|h-1} \]  \hspace{1cm} (1)
\[ P_{h|h-1} = F_{h-1}P_{h-1|h-1}F_{h-1}^T + Q_h \]  \hspace{1cm} (2)

Measurement update

\[ G_h = P_{h|h-1}H_h^T \left( H_hP_{h|h-1}H_h^T + R_h \right)^{-1} \]  \hspace{1cm} (3)
\[ X_{h|h} = X_{h|h-1} + G_h \left( Y_h - H_hX_{h|h-1} \right) \]  \hspace{1cm} (4)
\[ P_{h|h} = P_{h|h-1} - G_hH_hP_{h|h-1} \]  \hspace{1cm} (5)

**end for**
Extended Kalman Filter

Initialization

\[
X_{0|0} = X_0, \quad P_{0|0} = P_0
\]

\textbf{for} \ h = 1 \text{ to } N \textbf{ do}

Time update

\[
X_{h|h-1} = F_{h-1} X_{h-1|h-1}
\]

\[
P_{h|h-1} = F_{h-1} P_{h-1|h-1} F_{h-1}^T + Q_h
\]

Linearization

\[
H_h = \frac{\partial h(x^*)}{\partial x^*} \bigg|_{x^* = x_{h|h-1}}
\]

Measurement update

\[
G_h = P_{h|h-1} H_h^T \left( H_h P_{h|h-1} H_h^T + R_h \right)^{-1}
\]

\[
X_{h|h} = X_{h|h-1} + G_h \left( Y_h - H_h X_{h|h-1} \right)
\]

\[
P_{h|h} = P_{h|h-1} - G_h H_h P_{h|h-1}
\]

\textbf{end for}
Limiting Extended Kalman Filter (LimEKF)

Generation of limiting Kalman gain matrix $G$ (and $H$)

Initialization

$$X_{0|0} = X_0, \ P_{0|0} = P_0$$

for $h = 1$ to $N$ do

Time update

$$X_{h|h-1} = F_{h-1} X_{h-1|h-1}$$
$$P_{h|h-1} = F_{h-1} P_{h-1|h-1} F_{h-1}^T + Q_h$$

Measurement update

$$X_{h|h} = X_{h|h-1} + G [Y_h - h \ (X_{h|h-1})]$$
$$P_{h|h} = P_{h|h-1} - GHP_{h|h-1}$$

end for