Dynamic Traffic Assignment models and the problem of their calibration

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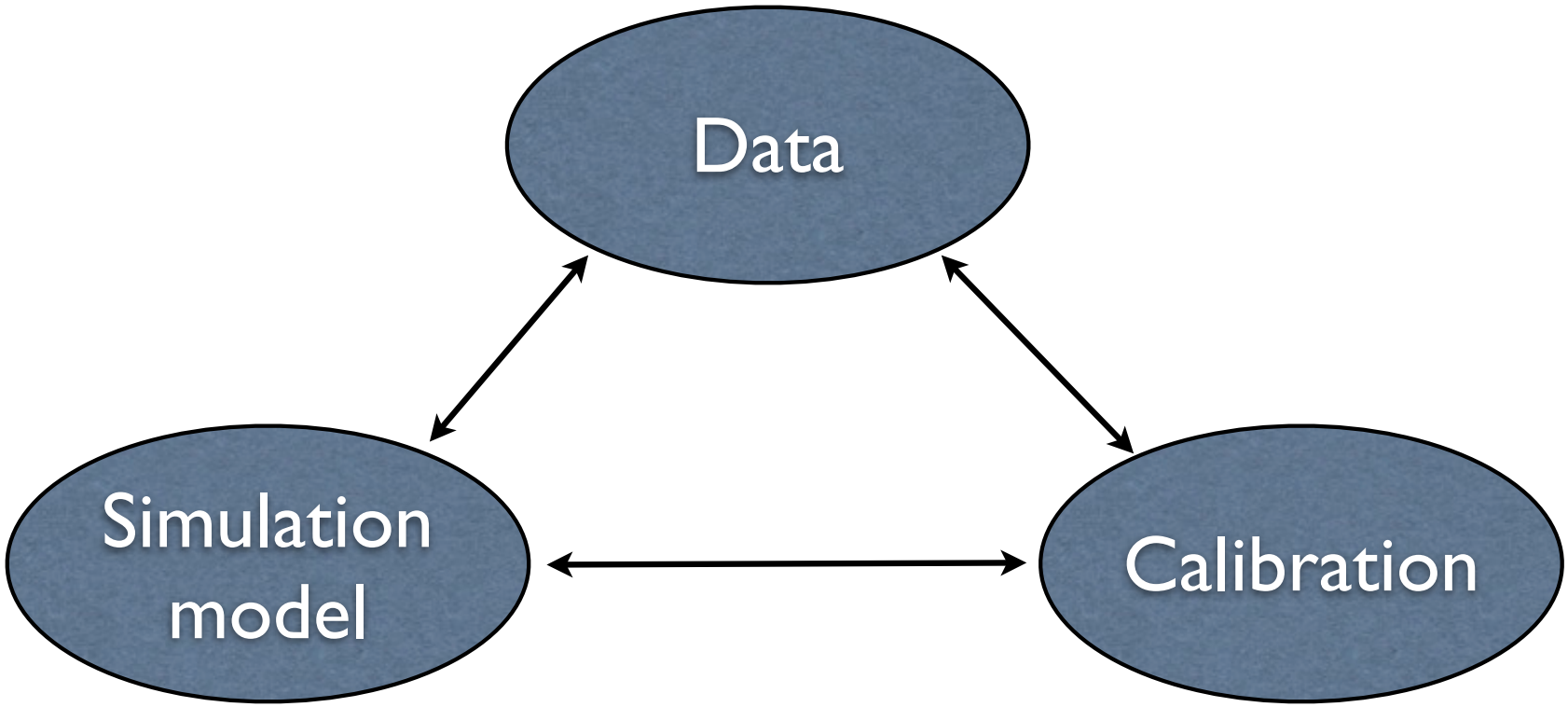
October 17, 2012

COST/Multitude Workshop within the framework of RelStat2012 - Riga, Latvia

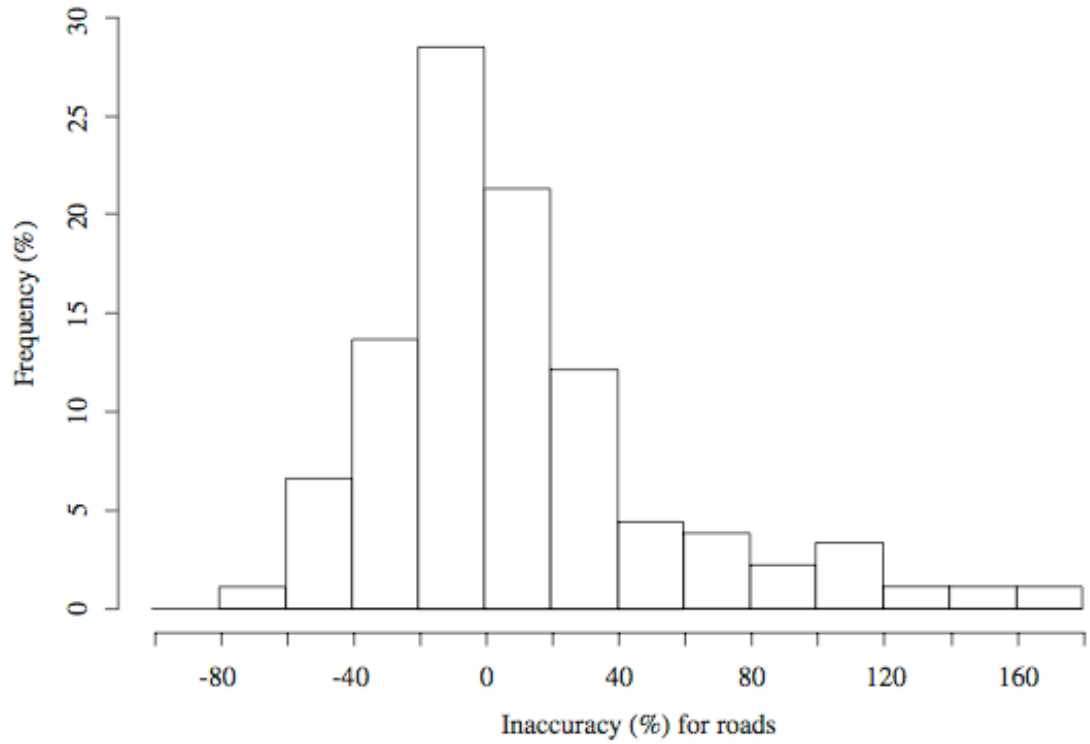
Outline

- Introduction / Motivation
- Calibration framework
  - Formulation: (i) general, (ii) off–line, (iii) on–line
  - Solution approaches
- Case studies
  - DTA, off–line, Los Angeles, CA and Lower Westchester County, NY
  - DTA, on–line, Southampton, UK
  - Microscopic traffic simulator, Lower Westchester County, NY
- Directions for further research

Motivation

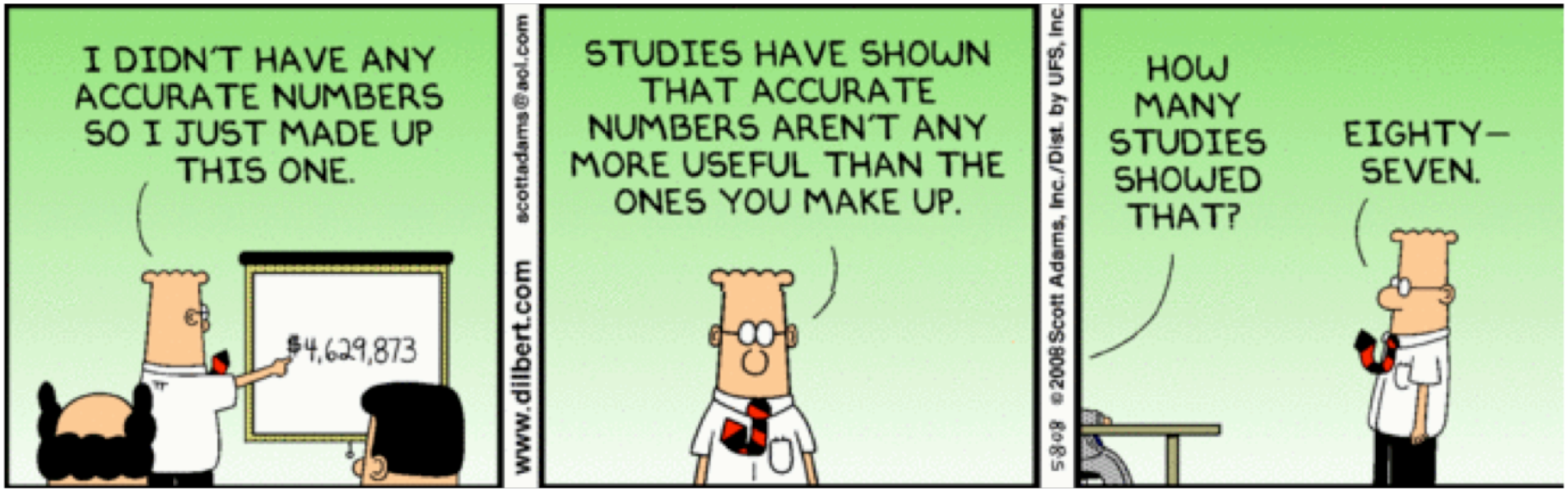


Motivation



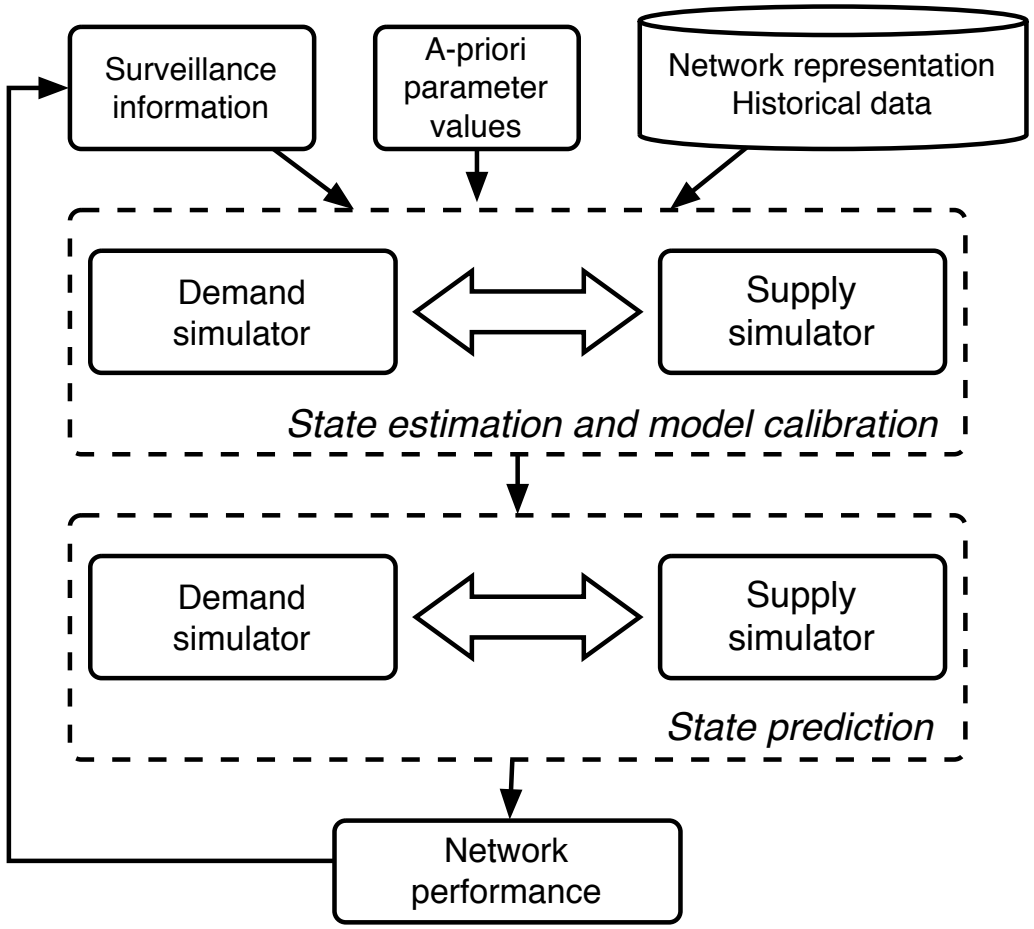
"For 50% of road projects, the difference between actual and forecasted traffic is more than 20%; for 25% of road projects, the difference is larger than 40%." (Source: Flyvbjerg, B., M. K. Skamris Holm, and S. L. Buhl (2006). Inaccuracy in traffic forecasts. Transport Reviews, Vol. 26, No. 1, 1-24, January 2006.

Motivation



Source: Dilbert.com

On–line DTA framework



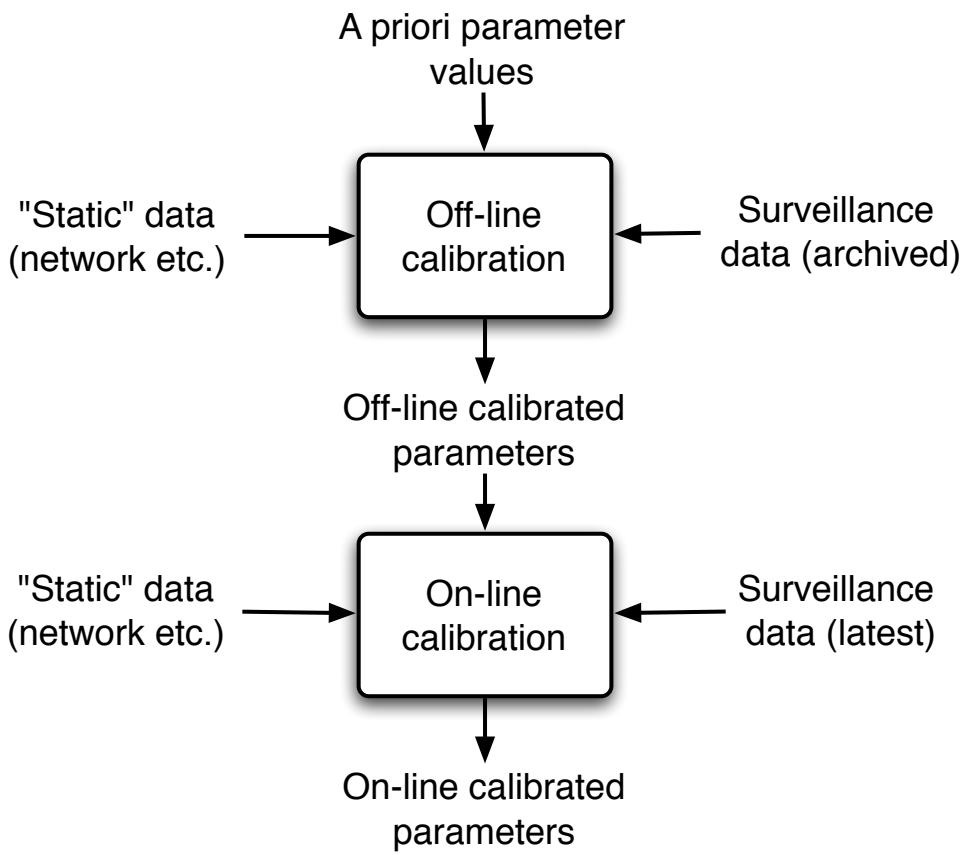
Motivation (seriously now)

- Models/components
  - Demand–side
    - OD flows
    - Behavioral models (e.g. route choice)
  - Supply–side
    - Speed–density relationship (e.g. $u = u_f \left[1 - \left(\frac{\max(0, K - K_{min})}{K_{jam}}\right)^{\beta}\right]^{\alpha}$
      - where u denotes the speed, u_f is the free flow speed, K is the density, K_{min} is the minimum density, K_{jam} is the *jam* density and α and β are model parameters.)
    - Segment capacities
- Often, state estimation and model calibration approaches include subset of these models

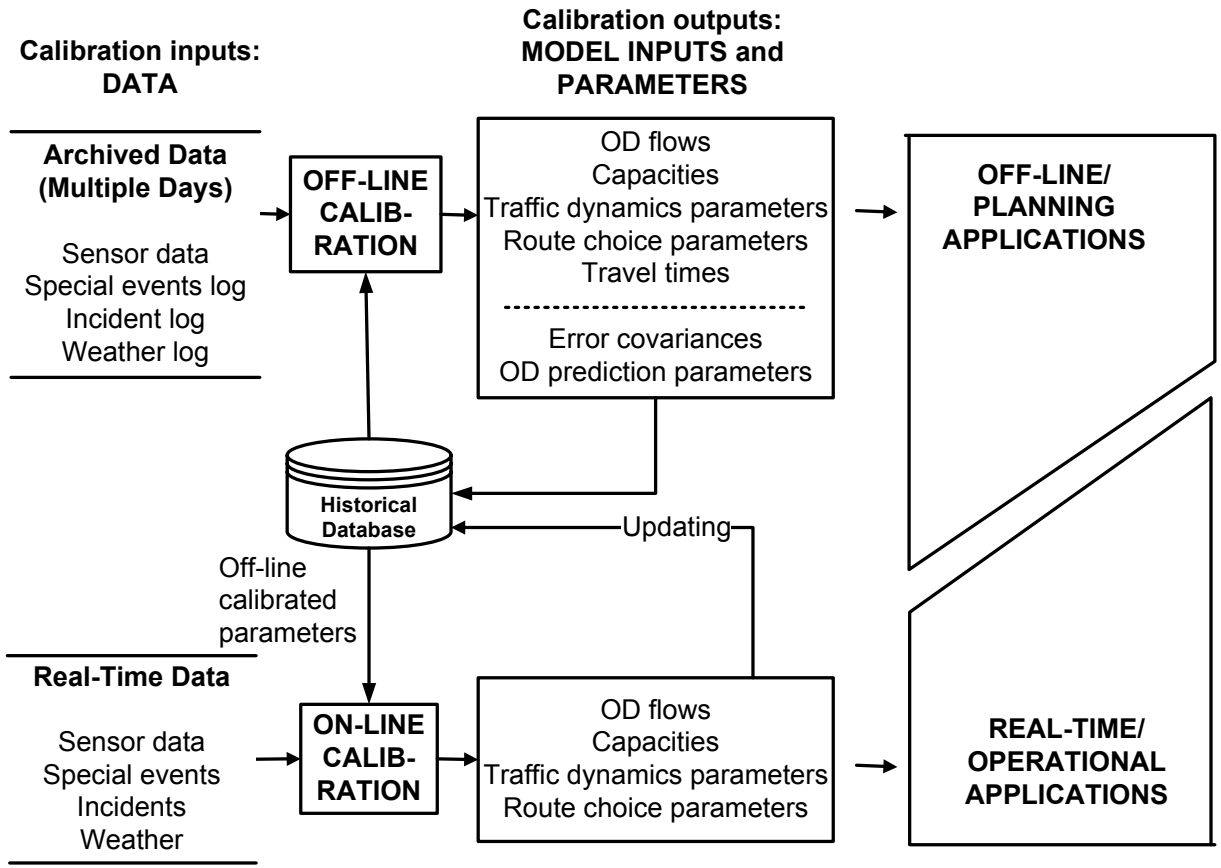
Research objectives

- Develop an integrated approach for calibration that
  - Jointly estimates and predicts demand and supply parameters
  - Is general and flexible
    - Not tied to a particular simulation system
    - Applicable to any calibration parameters
    - Can exploit any available information
  - Is computationally feasible
- Demonstrate the feasibility of the approach

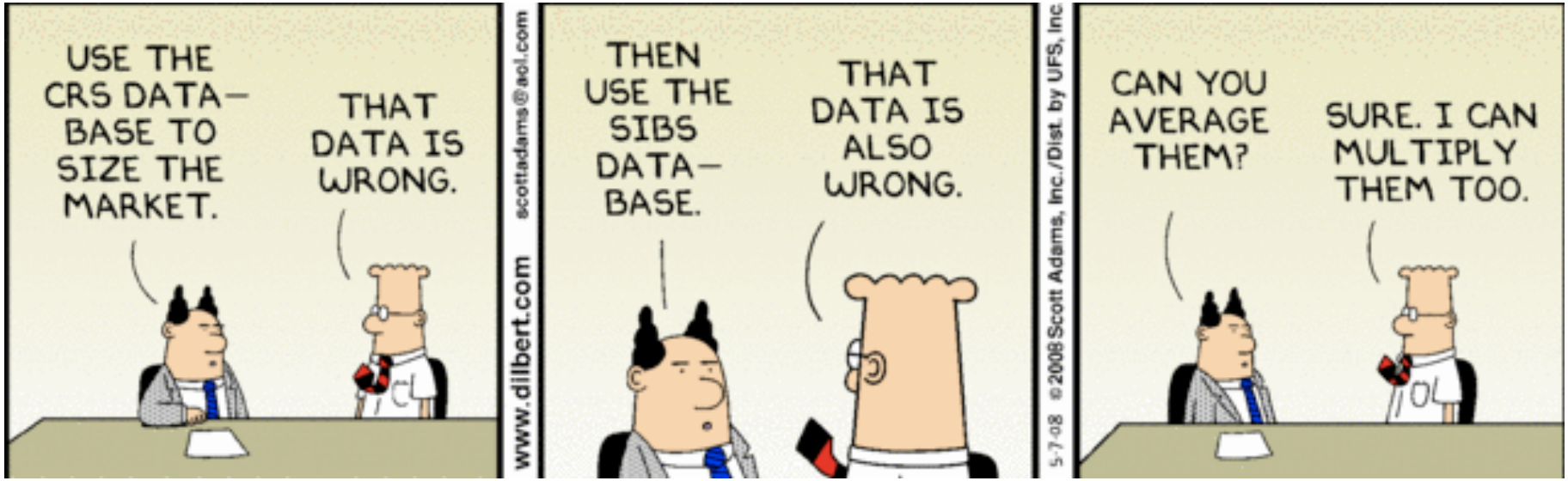
Inputs and outputs



Integrated calibration framework



Motivation



Source: Dilbert.com

Integrated calibration framework

A priori estimates of model inputs and parameters, if available, contain valuable structural information about the calibration variables and represent **direct measurements**:

$$\begin{aligned} \mathbf{x}^{a} &= \mathbf{x} + \mathbf{v} \\ \boldsymbol{\beta}^{a} &= \boldsymbol{\beta} + \mathbf{w} \end{aligned}$$

$\mathbf{x}$ and $\boldsymbol{\beta}$ are vectors of the OD flows and model parameters to be calibrated respectively; $\mathbf{x}^{a}$ and $\boldsymbol{\beta}^{a}$ are the corresponding a priori values, $\mathbf{v}$ and $\mathbf{w}$ are error vectors.

Integrated calibration framework

Formulation as an **optimization problem**

$$(\boldsymbol{\beta}^*, \mathbf{x}^*) = \underset{\boldsymbol{\beta},\mathbf{x}}{\arg\min} z(\mathbf{M}, \mathcal{M}, \mathbf{x}, \mathbf{x}^a, \boldsymbol{\beta}, \boldsymbol{\beta}^a)$$

In the above expression, $\mathbf{x}_h$ is the vector of OD flows departing in interval h; $\mathbf{x}_h^a$ are initial OD flow estimates; $\boldsymbol{\beta}_h$ is a vector of route choice and supply model parameters; $\boldsymbol{\beta}_h^a$ are the corresponding initial values; $\mathbf{x} = \{\mathbf{x}_h | h \in \tau\}$; $\mathbf{x}^a = \{\mathbf{x}_h^a | h \in \tau\}$; $\boldsymbol{\beta} = \{\boldsymbol{\beta}_h | h \in \tau\}$; $\boldsymbol{\beta}^a = \{\boldsymbol{\beta}_h^a | h \in \tau\}$.

Off–line vs. on–line calibration

- Off–line calibration is a global optimization step. On-line calibration, on the other hand, attempts to dynamically adjust the off-line calibrated parameter values

- Large archived databases of prior observations may be available for the off-line calibration while on-line calibration uses the most recent time period

- While off-line calibration has no explicit computational requirements, on-line calibration has to be performed in real time

- Off-line calibration must involve the estimation of a complete set of simulation model inputs and parameters, while on-line calibration may focus on a subset of parameters

Off–line calibration formulation

Formulated as an optimization problem

$$
\begin{aligned}
(\boldsymbol{\beta}^*, \mathbf{x}^*) &= \underset{\boldsymbol{\beta},\mathbf{x}}{\arg\min} z \\
&= \sum_{h=1}^{H} [z_1(\mathbf{M}_h, \mathcal{M}_h,) + z_2(\mathbf{x}_h, \mathbf{x}_h^a) + z_3(\boldsymbol{\beta}_h, \boldsymbol{\beta}_h^a)]
\end{aligned}
$$

subject to:

$$
\mathbf{M}_h = f(\mathbf{x}_1, \ldots, \mathbf{x}_h, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_h; G_1, \ldots, G_h)
$$

$$
l_h^x \leq \mathbf{x}_h \leq u_h^x
$$

$$
l_h^\beta \leq \boldsymbol{\beta}_h \leq u_h^\beta
$$

Off–line calibration — Solution approaches

- Path search
  - Stochastic Approximation
    - Finite Difference Stochastic Approximation
    - Simultaneous Perturbation Stochastic Approximation
  - Response Surface Methodology
    - Involves local polynomial approximation of the surface at each iteration, extensions, such as SNOBFIT
- Pattern search, e.g. Hooke and Jeeves, Nelder and Mead's Downhill Simplex
- Random search methods, including metaheuristics such as
  - Genetic Algorithms
  - Simulated Annealing

Off–line calibration — Solution approaches: SPSA

- Simultaneous Perturbation Stochastic Approximation
  - Spall (1992), Kunde (2002), Balakrishna (2006), Balakrishna et al. (2006), Balakrishna et al. (2007)
  - Iteratively updates θ

$$\theta^{i+1} = \theta^{i} - a^{i}\hat{g}^{i}\left(\theta^{i}\right)$$

  - Gradient approximation

$$\hat{g}_{k}^{i} = \frac{z\left(\theta^{i} + c^{i}\Delta_{i}\right) - z\left(\theta^{i} - c^{i}\Delta_{i}\right)}{2c^{i}\Delta_{ik}}$$

where $\theta = [\mathbf{x}; \boldsymbol{\beta}]$

On–line calibration formulation

- State–space model
  - State vector
  - Measurement equations
  - Transition equations
- Idea of deviations

State vector

- Deviations $\Delta\boldsymbol{\pi}_h$ of model inputs and parameters $\boldsymbol{\pi}_h$ from available estimates

$$\Delta\boldsymbol{\pi}_h = \boldsymbol{\pi}_h - \boldsymbol{\pi}_h^H$$

- State vector includes
  - OD flows x_{ijh}: number of vehicles departing from origin i towards destination j during time interval h
  - Model parameters, e.g.
    - route choice model parameters
    - speed–density relationship parameters
    - segment capacities

Measurement equations (I)

- Direct measurement equation

$$\boldsymbol{\pi}_h^a = \boldsymbol{\pi}_h + \mathbf{v}_h$$

- In deviations' form (subtracting $\boldsymbol{\pi}_h^H$ from both sides)

$$\begin{aligned} \boldsymbol{\pi}_h^a - \boldsymbol{\pi}_h^H &= \boldsymbol{\pi}_h - \boldsymbol{\pi}_h^H + \mathbf{v}_h \Rightarrow \\ \Delta\boldsymbol{\pi}_h^a &= \Delta\boldsymbol{\pi}_h + \mathbf{v}_h \end{aligned}$$

  where

  – a denotes *a priori* values, H indicates historical estimates, $\mathbf{v}_h$ is a random error vector.

Measurement equations (II)

- Indirect measurement equation

$$\mathbf{M}_h = \mathcal{S}\big(\boldsymbol{\pi}_h\big) + \boldsymbol{\upsilon}_h$$

- In deviations' form (subtracting $\mathbf{M}_h^H$ from both sides)

$$\begin{aligned} \mathbf{M}_h - \mathbf{M}_h^H &= \mathcal{S}\big(\boldsymbol{\pi}_h\big) - \mathbf{M}_h^H + \boldsymbol{\upsilon}_h \Rightarrow \\ \Delta\mathbf{M}_h &= \mathcal{S}\big(\boldsymbol{\pi}_h^H + \Delta\boldsymbol{\pi}_h\big) - \mathbf{M}_h^H + \boldsymbol{\upsilon}_h \end{aligned}$$

where

- $\mathbf{M}_h$ is a vector of measurements for interval h, and $\boldsymbol{\upsilon}_h$ is a random error vector.
- $\mathcal{S}$ is the simulator function mapping the state to the measurements (no analytic expression).

Transition equations

$$\boldsymbol{\pi}_{h+1} = \sum_{q=h-p}^{h} \mathcal{F}_q^{h+1} \cdot \boldsymbol{\pi}_q + \boldsymbol{\eta}_h$$

- In deviations' form (subtracting $\boldsymbol{\pi}^H$ for appropriate interval from both sides)

$$\begin{aligned} \boldsymbol{\pi}_{h+1} - \boldsymbol{\pi}_{h+1}^H &= \sum_{q=h-p}^{h} \mathcal{F}_q^{h+1} \left(\boldsymbol{\pi}_q - \boldsymbol{\pi}_q^H\right) + \boldsymbol{\eta}_h \Rightarrow \\ \Delta\boldsymbol{\pi}_{h+1} &= \sum_{q=h-p}^{h} \mathcal{F}_q^{h+1} \cdot \Delta\boldsymbol{\pi}_q + \boldsymbol{\eta}_h \end{aligned}$$

  where
  - $\mathcal{F}_q^{h+1}$ is a transition matrix of effects of $\Delta\boldsymbol{\pi}_q$ on $\Delta\boldsymbol{\pi}_{h+1}$
  - p is the degree of the autoregressive process, and
  - $\boldsymbol{\eta}_h$ is a random error vector

The model at a glance

$$\begin{aligned}\Delta\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{a}} &= \Delta\boldsymbol{\pi}_{\mathrm{h}} + \mathbf{v}_{\mathrm{h}} \\ \Delta\mathbf{M}_{\mathrm{h}} &= \mathcal{S}\big(\boldsymbol{\pi}_{\mathrm{h}}^{\mathrm{H}} + \Delta\boldsymbol{\pi}_{\mathrm{h}}\big) - \mathbf{M}_{\mathrm{h}}^{\mathrm{H}} + \boldsymbol{\upsilon}_{\mathrm{h}}\end{aligned}$$

$$\Delta\boldsymbol{\pi}_{\mathrm{h+1}} = \sum_{\mathrm{q=h-p}}^{\mathrm{h}} \mathcal{F}_{\mathrm{q}}^{\mathrm{h+1}} \cdot \Delta\boldsymbol{\pi}_{\mathrm{q}} + \boldsymbol{\eta}_{\mathrm{h}}$$

(Or, equivalently, as an optimization formulation:)

$$\pi_h^* = argmin[N_1(e_h^a) + N_2(e_h^{obs}) + N_3(e_h^{auto})]$$

$$= argmin[N_1(\pi_h - \pi_h^a) + N_2(M_h^{sim} - M_h^{obs}) + N_3(\pi_h - \sum_{q=h-p-1}^{h-1} F_q^{h+1}\pi_q)]$$

Solution approaches

- Linear models
  - Kalman Filter (KF)
- Non–linear models
  - Extended Kalman Filter (EKF)
    - Need numerical derivatives if no analytical expression
  - Limiting Extended Kalman Filter (LimEKF)
    - Use off-line pre-computed Kalman gain
    - Tomorrow: interesting results using different ways to select/compute the Kalman gain
  - Unscented Kalman Filter (UKF)
    - Same computational complexity as EKF
  - SP-EKF: Extended Kalman Filter using SPSA for numerical derivative

Kalman Filtering principles

- Prediction–correction approach

1. **Predict** *a priori* state for interval h (using information up to $h-1$)

2. Compute *Kalman gain* matrix

3. Combine Kalman gain and surveillance from interval h to **correct** state prediction for interval h

Extended Kalman Filter

- Extension of Kalman Filter for non–linear models
  - First order Taylor expansion approximation
- Can use numerical derivatives, if no analytical relationship exists (for measurement equations)
  - Large number of function evaluations is required
  - e.g. $2n$ evaluations if central differences are used (n is state dimension)
- Computation of *Kalman gain* most "expensive" operation

Limiting Extended Kalman Filter

- Use a pre–computed Kalman gain
  - E.g.: (weighted) average of Kalman gains (computed off–line)
  - No need to compute the Kalman gain on–line
  - Tomorrow: interesting results using different ways to select/compute the Kalman gain
- Only a single function evaluation is required on–line
- Can run EKF off–line and periodically re–compute Kalman gain

Case studies — Objectives

- Demonstrate feasibility of the calibration approach
- Verify importance of calibration
  - Impact of joint estimation and prediction of demand and supply parameters
- Test candidate algorithms based on several criteria
  - Performance/Fit (estimation and prediction)
  - Computational properties
  - Robustness

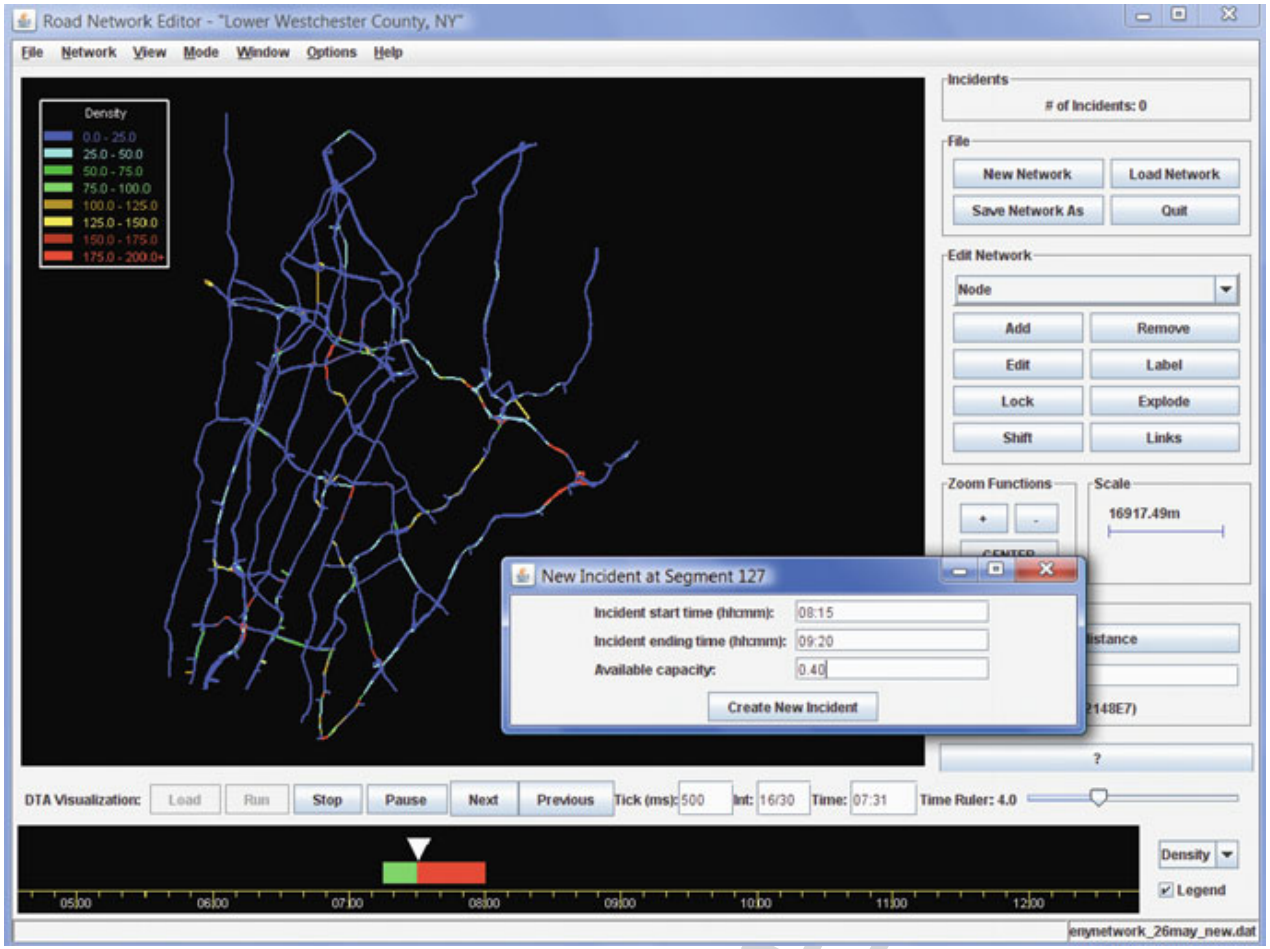
Case studies

- DTA, off–line calibration
  - (Downtown) Los Angeles, CA
  - Lower Westchester County, NY (with synthetic AVI data)
- DTA, on–line calibration
  - Southampton, UK
- Microscopic traffic simulator
  - Lower Westchester County, NY

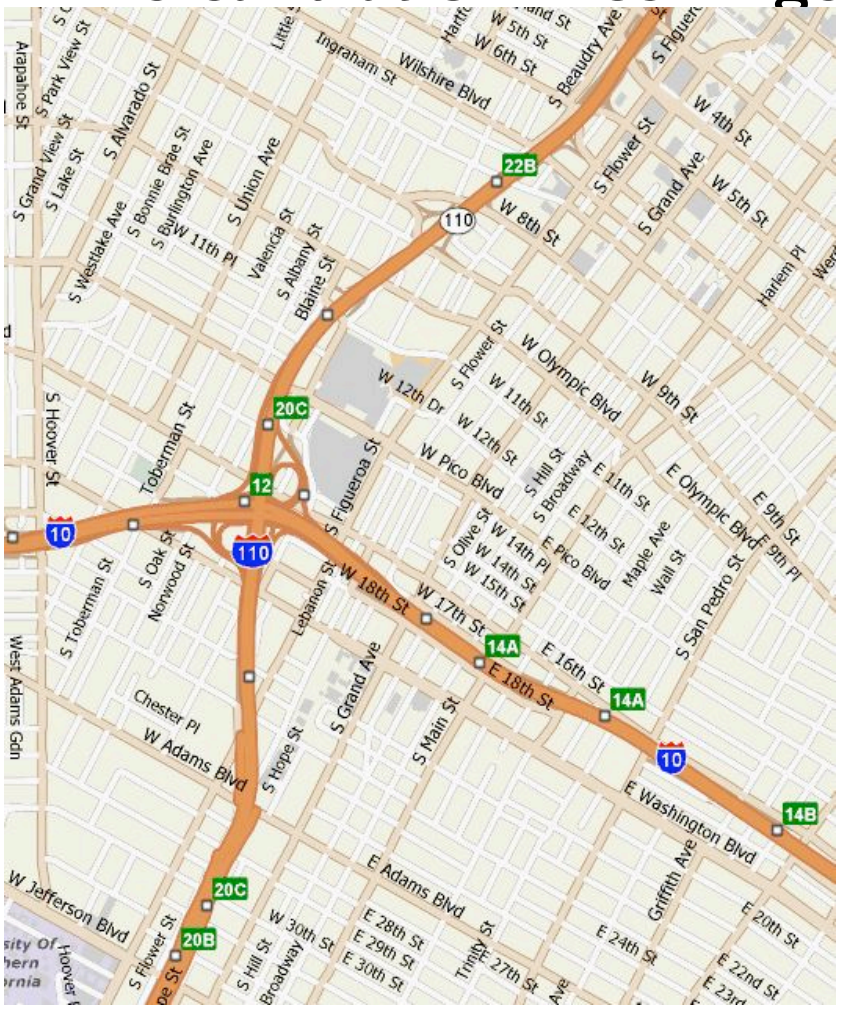
The DynaMIT–R system

- State–of–the–art, simulation–based DTA system
  - Transportation network and supply characteristics
  - OD demand and travel behavior
- Operates in rolling horizon for:
  - Real–time estimation of network performance
  - Short–term prediction of future network conditions
  - Generation of traffic information
- Inputs
  - A-priori OD flows
  - Route choice parameters
  - Traffic dynamics models' parameters
  - Segment capacities

DynaMIT–R GUI



Off–line calibration: Los Angeles



Off–line calibration: Los Angeles

- 243 nodes / 606 links
- Archived freeway and arterial flow and speed data from 203 loop detectors
- 3:00 – 9:00 a.m. (divided into 15-min intervals)
- Fairly congested network with a large number of route choices for each of the 1129 OD pairs.
- Time- dependent OD flows were calibrated for each interval, together with route choice model parameters, segment capacities, and speed–density function parameters.

Off–line calibration (LA) — Case study results

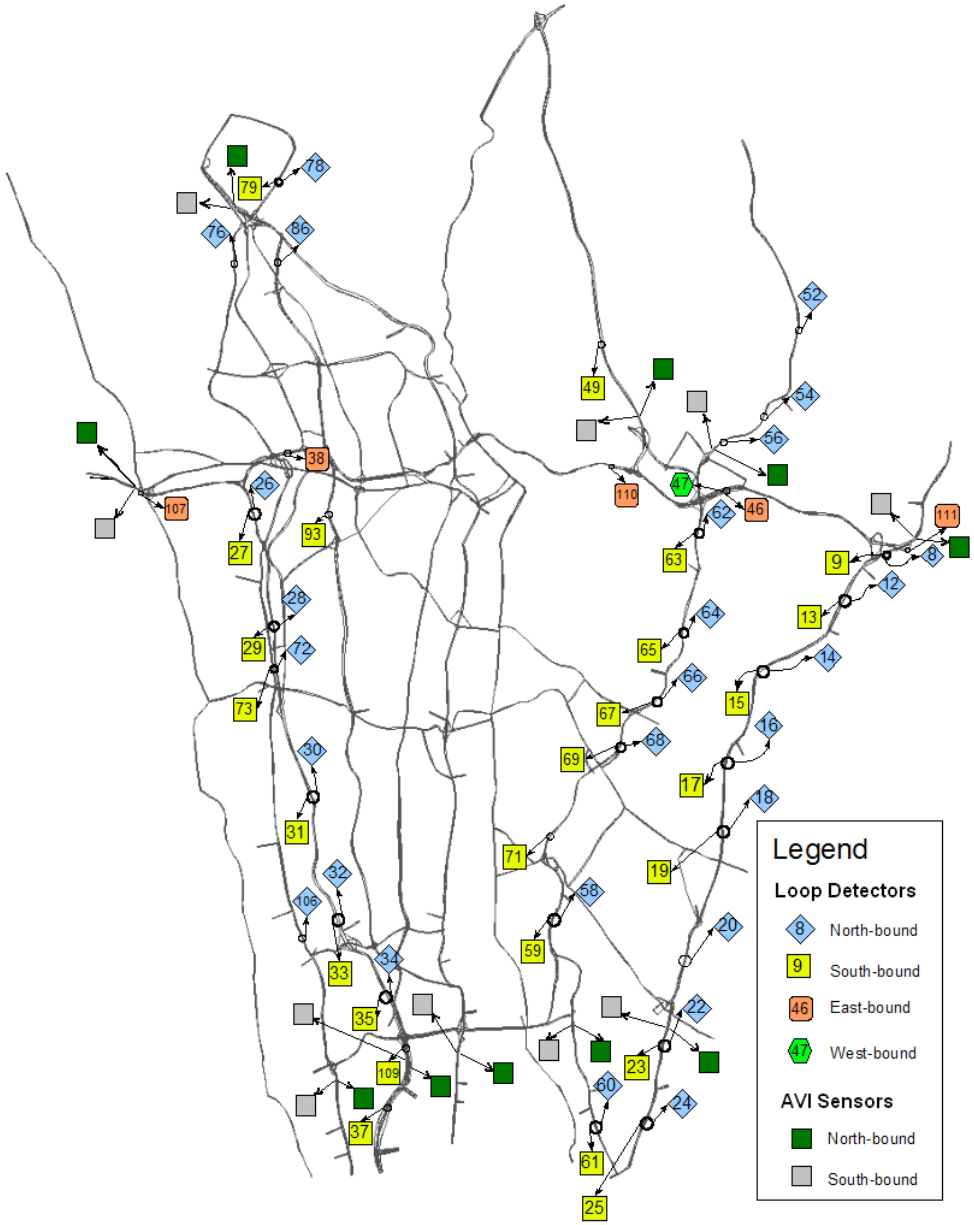
The performance of the calibration methodology was assessed using the normalized root mean square error (RMSN) of the counts and speeds:

$$\mathrm{RMSN} = \frac{\sqrt{N \sum_N (y - \hat{y})^2}}{\sum_N y}$$

where N is the number of measurements, y denotes the observed, and $\hat{y}$ the estimated traffic parameters, e.g. counts or speeds.

Table 10.2 Fit to counts and speeds, RMSN (off-line calibration)

	Fit to counts		Fit to speeds	
Estimator	Freeway	Arterial	Freeway	Arterial
Reference	0.218	0.239	0.181	0.203
D	0.114	0.143	0.118	0.125
SD	0.090	0.113	0.088	0.093


Legend
Loop Detectors
North-bound
South-bound
East-bound
West-bound
AVI Sensors
North-bound
South-bound

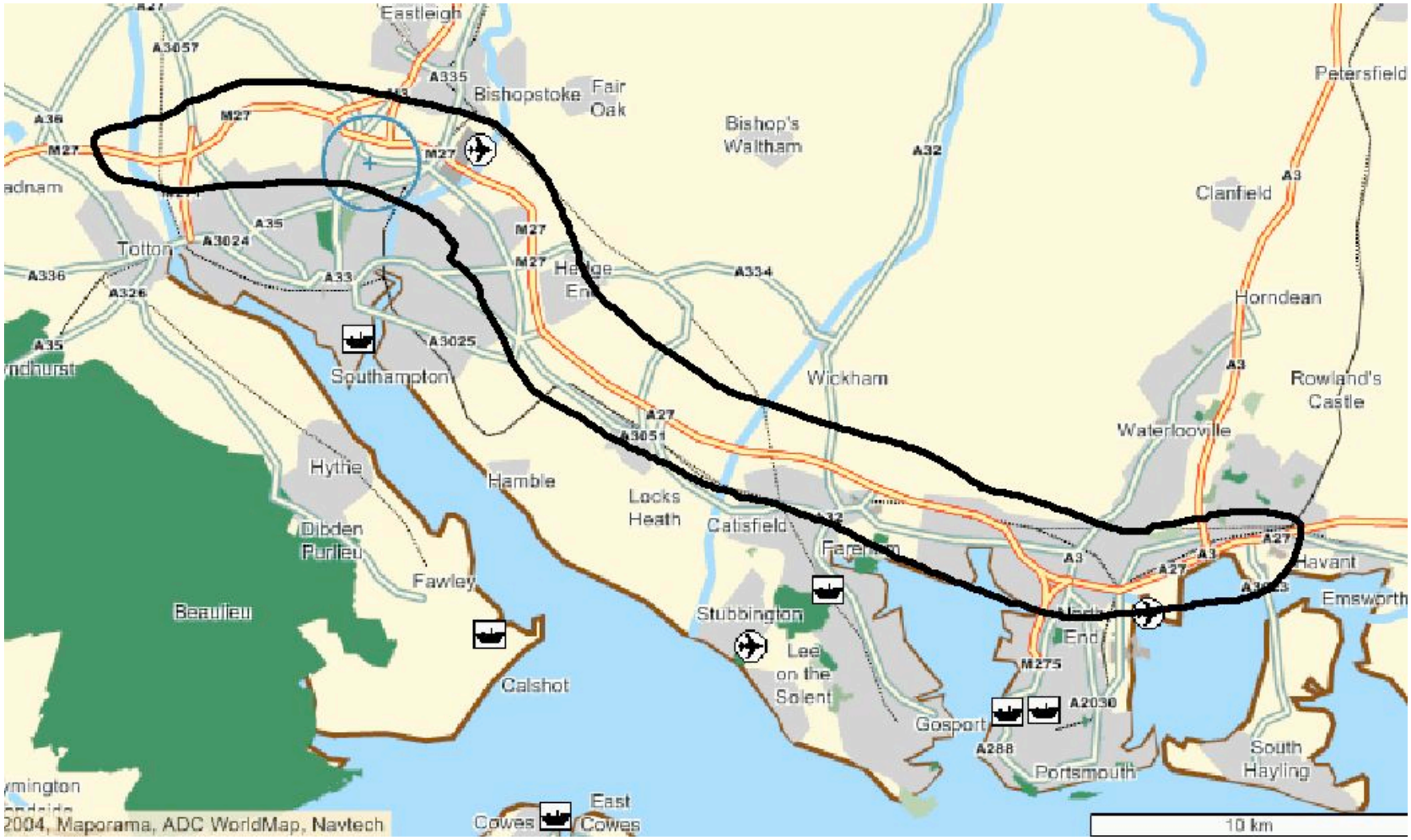
DTA, Off–line calibration – LWC

- 1767 links (split into 2564 segments) / 825 nodes / 482 origin-destination pairs / 55 point data locations / 10 AVI locations
- 6:00 am to 8:00 am divided into eight 15-minute intervals
- Total number of parameters for the case of joint demand-supply calibration is 3856 OD flows plus 2564 segment capacities plus 50 speed-density parameters equals 6470 parameters.

Table 10.4 Calibration and validation results for Lower Westchester County network

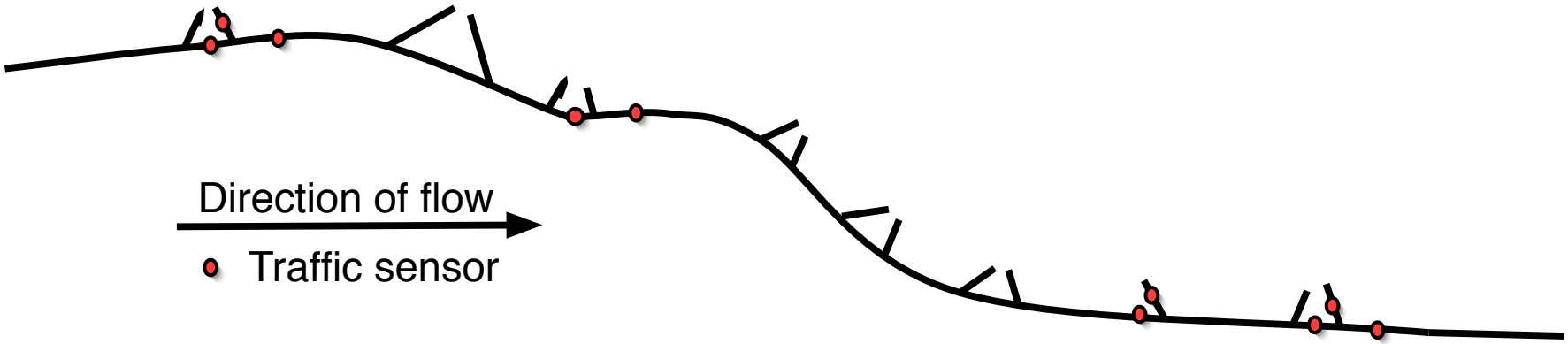
	Demand-only calibration		Joint demand-supply calibration		Joint demand–supply validation
RMSN	Link counts (%)	Counts + AVI (%)	Link counts (%)	Counts + AVI (%)	Counts + AVI (%)
Counts	20.09	18.24	20.69	17.94	17.63
Travel Times	22.23	21.24	19.46	18.92	18.55

DTA, On–line calibration – Southampton, UK



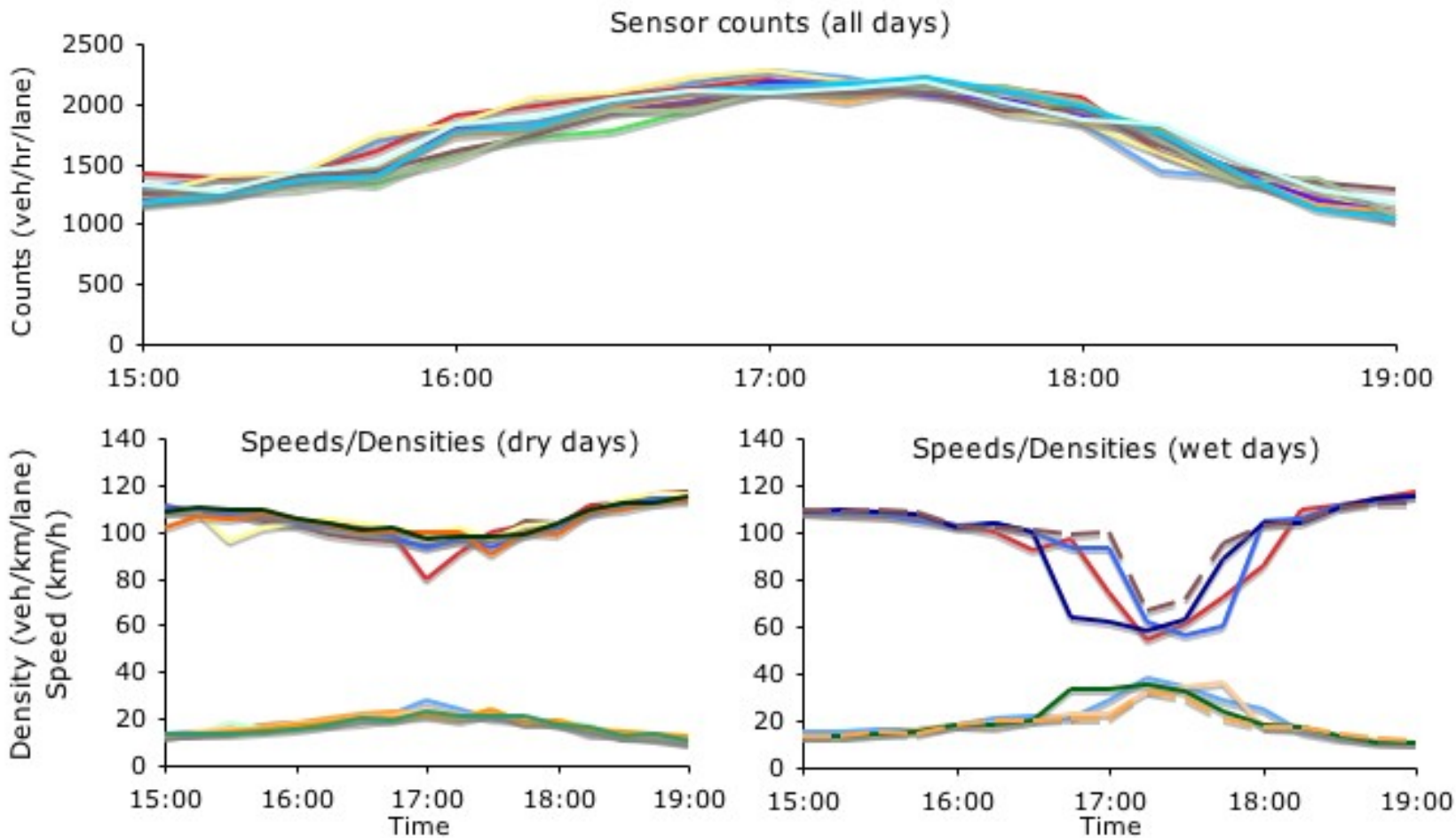
Southampton, UK — Network

Southampton, U.K. (35km portion of freeway M27)

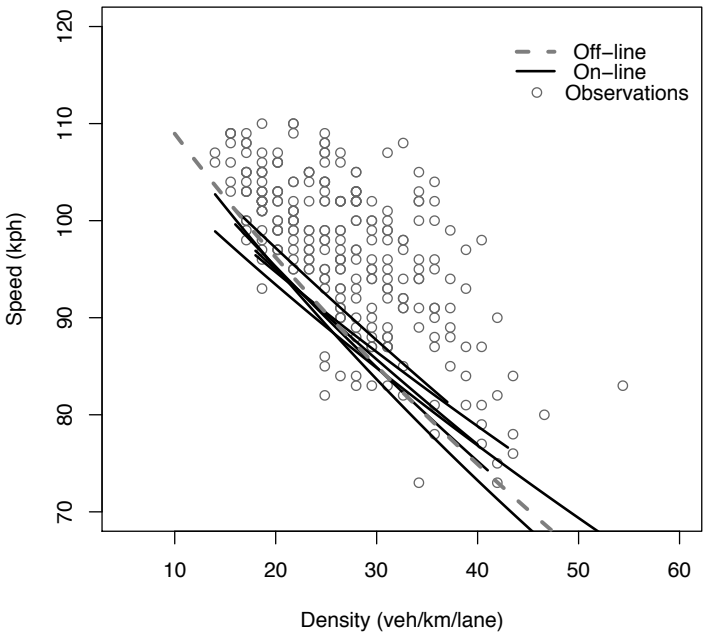


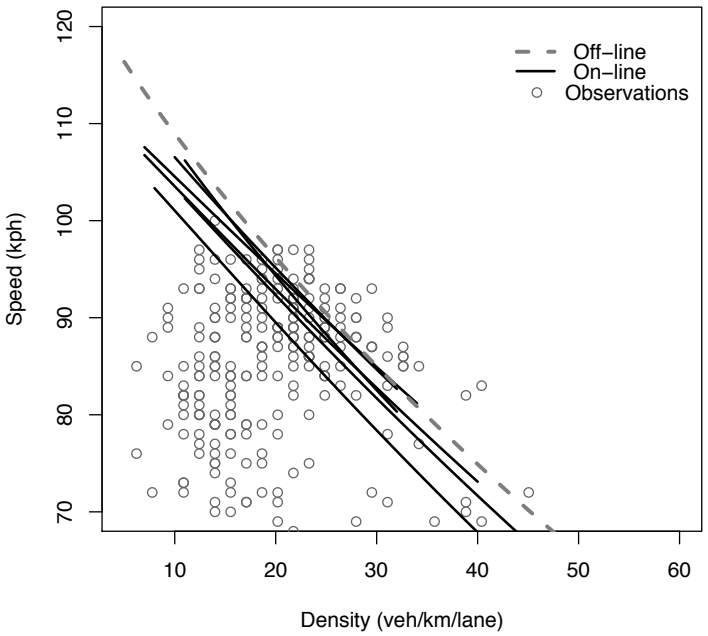
- 45 segments (three types)
- 10 sensors
- 20 OD pairs
- PM data (peak direction)

Traffic flow characteristics



On–line calibrated speed–density relationships
Mainline segments – EKF
top: dry weather, bottom: wet weather





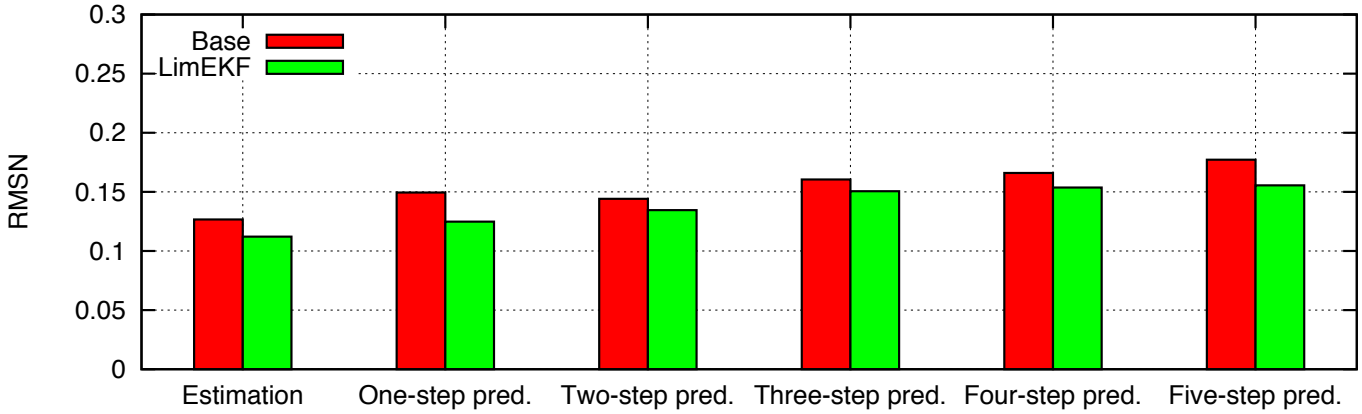
On–line calibration results

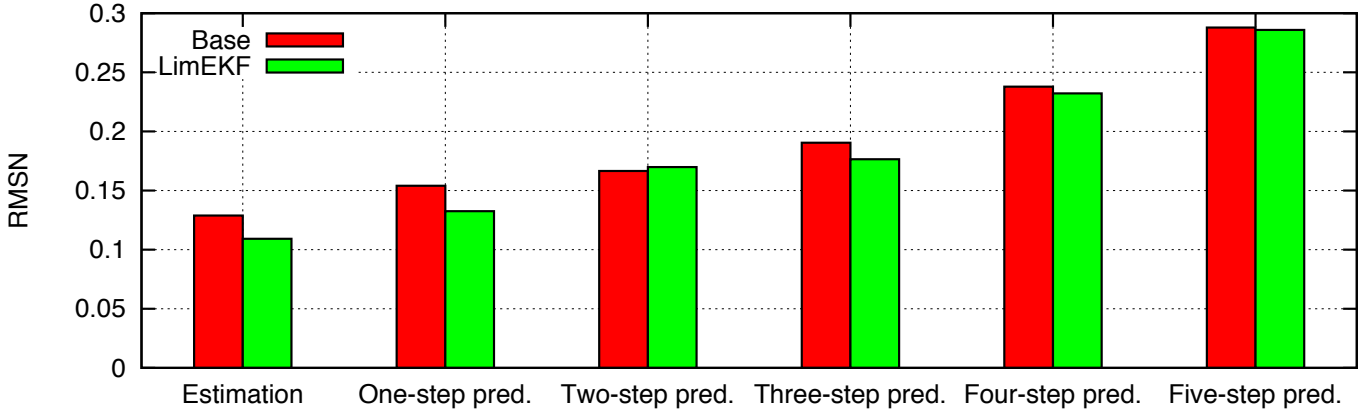
Table 10.3 Fit to counts and speeds

	Speeds		
	Estimated	One-step predicted	Two-step predicted
Online (EKF)	0.1106	0.1209	0.1303
Online (LimEKF)	0.1120	0.1249	0.1347
	Counts		
	Estimated	One-step predicted	Two-step predicted
Online (EKF)	0.1039	0.1318	0.1550
Online (LimEKF)	0.1091	0.1321	0.1702

(Note: LimEKF robustness)

On–line calibration — Case study results LimEKF (top:speeds, bottom:counts)





Computational performance

- EKF
  - $n + 1$ or $2n$ function evaluations (essentially forward or central numerical derivative, respectively)
- Limiting Kalman Filter
  - O(1) computational performance
  - Single function evaluation required (independent of state dimension)

Off–line calibration — Microscopic traffic simulator

- Difference between various levels of simulation (micro-meso-macro)
  - Number of parameters / problem dimension
  - Stochasticity
- Demonstrate that the approach is flexible/robust
  - Framework
  - Solution approaches
- Case study setup
  - MITSIMLab (open source version available at: http://web.mit.edu/its/MITSIMLabOSnew.html)
  - Lower Westchester County network
  - Solution approach: SPSA

Off–line calibration — Microscopic Traffic Simulator

Time	RMSN			RMSPE			MPE		
Period	Before	After	% change	Before	After	% change	Before	After	% change*
6:00 – 6:15	0.537	0.179	67%	0.535	0.216	60%	-0.474	0.004	99%
6:15 – 6:30	0.219	0.158	28%	0.224	0.172	23%	-0.128	-0.006	95%
6:30 – 6:45	0.208	0.155	26%	0.190	0.148	22%	-0.137	-0.044	68%
6:45 – 7:00	0.263	0.198	25%	0.253	0.181	29%	-0.171	-0.052	70%
7:00 – 7:15	0.405	0.248	39%	0.381	0.225	41%	-0.271	-0.121	55%
7:15 – 7:30	0.528	0.245	54%	0.443	0.237	47%	-0.335	-0.118	65%
7:30 – 7:45	0.553	0.236	57%	0.454	0.217	52%	-0.353	-0.120	66%
7:45 – 8:00	0.546	0.160	71%	0.453	0.157	65%	-0.337	-0.062	82%
8:00 – 8:15	0.656	0.236	64%	0.638	0.230	64%	-0.614	-0.070	89%
8:15 – 8:30	0.435	0.249	43%	0.390	0.242	38%	-0.262	-0.065	75%
8:30 – 8:45	0.427	0.285	33%	0.409	0.294	28%	-0.199	-0.011	94%
8:45 – 9:00	0.420	0.268	36%	0.403	0.281	30%	-0.192	0.030	84%
6:00 – 9:00	*0.476*	*0.231*	*51%*	*0.416*	*0.221*	*47%*	*-0.289*	*-0.053*	*82%*

* in absolute values

Primary benefits of the approach

- Non-linear dependencies between traffic data and calibration variables are accurately captured
  - Consequently, linear approximations such as the assignment matrix mapping are not required.
- All demand and supply parameters ($\mathbf{x}_h$ and $\boldsymbol{\beta}_h$) may be estimated simultaneously and efficiently
- Any general traffic data can be included in $\mathbf{M}_h$
- Capacity reduction factors can be estimated for known incidents in the observed data, to accurately incorporate supply impacts on calibration
- Supply parameters for links without sensors can be updated independently, using the parameters from the closest links only as starting values.

Directions for further research

- Further experimental analysis
  - Scalability
  - Robustness
  - Sensitivity
- Selection of parameters (primarily for on–line calibration)
  - Dynamic (?)
- Impact of grouping segments (for speed–density relationship estimation)
- Algorithms
  - e.g. particle filters (generalization of UKF)
  - parallel versions of algorithms

"Controversial" point/question

What are the boundaries between off–line and on–line calibration?

- Do we need both?
- Given a reasonable warm–up period, do we need off–line calibration?
- Unless there are disruptions, is on–line calibration a waste of time?

Indicative references

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Thank you

Questions?

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Backup slides

Kalman Filter

Initialization

$$\mathbf{X}_{0|0} = \mathbf{X}_0$$
$$\mathbf{P}_{0|0} = \mathbf{P}_0$$

for $h = 1$ to N **do**

Time update

$$\mathbf{X}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{X}_{h-1|h-1} \tag{1}$$

$$\mathbf{P}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{P}_{h-1|h-1}\mathbf{F}_{h-1}^{T} + \mathbf{Q}_h \tag{2}$$

Measurement update

$$\mathbf{G}_h = \mathbf{P}_{h|h-1}\mathbf{H}_h^{T}\left(\mathbf{H}_h\mathbf{P}_{h|h-1}\mathbf{H}_h^{T} + \mathbf{R}_h\right)^{-1} \tag{3}$$

$$\mathbf{X}_{h|h} = \mathbf{X}_{h|h-1} + \mathbf{G}_h\left(\mathbf{Y}_h - \mathbf{H}_h\mathbf{X}_{h|h-1}\right) \tag{4}$$

$$\mathbf{P}_{h|h} = \mathbf{P}_{h|h-1} - \mathbf{G}_h\mathbf{H}_h\mathbf{P}_{h|h-1} \tag{5}$$

end for

Extended Kalman Filter

Initialization

$$\mathbf{X}_{0|0} = \mathbf{X}_0, \quad \mathbf{P}_{0|0} = \mathbf{P}_0$$

for $h = 1$ to N **do**

Time update

$$\mathbf{X}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{X}_{h-1|h-1} \tag{6}$$

$$\mathbf{P}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{P}_{h-1|h-1}\mathbf{F}_{h-1}^{T} + \mathbf{Q}_h \tag{7}$$

Linearization

$$\mathbf{H}_h = \left.\frac{\vartheta \mathbf{h}(\mathbf{x}^*)}{\vartheta \mathbf{x}^*}\right|_{\mathbf{x}^*=\mathbf{X}_{h|h-1}} \tag{8}$$

Measurement update

$$\mathbf{G}_h = \mathbf{P}_{h|h-1}\mathbf{H}_h^{T}\left(\mathbf{H}_h\mathbf{P}_{h|h-1}\mathbf{H}_h^{T} + \mathbf{R}_h\right)^{-1} \tag{9}$$

$$\mathbf{X}_{h|h} = \mathbf{X}_{h|h-1} + \mathbf{G}_h\left(\mathbf{Y}_h - \mathbf{H}_h\mathbf{X}_{h|h-1}\right) \tag{10}$$

$$\mathbf{P}_{h|h} = \mathbf{P}_{h|h-1} - \mathbf{G}_h\mathbf{H}_h\mathbf{P}_{h|h-1} \tag{11}$$

end for

Limiting Extended Kalman Filter (LimEKF)

Generation of *limiting* Kalman gain matrix $\mathbf{G}$ (and $\mathbf{H}$)

Initialization

$$\mathbf{X}_{0|0} = \mathbf{X}_0, \ \mathbf{P}_{0|0} = \mathbf{P}_0$$

for $h = 1$ to N **do**

Time update

$$\mathbf{X}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{X}_{h-1|h-1} \tag{12}$$

$$\mathbf{P}_{h|h-1} = \mathbf{F}_{h-1}\mathbf{P}_{h-1|h-1}\mathbf{F}_{h-1}^{T} + \mathbf{Q}_h \tag{13}$$

Measurement update

$$\mathbf{X}_{h|h} = \mathbf{X}_{h|h-1} + \mathbf{G}\left[\mathbf{Y}_h - \mathbf{h}\left(\mathbf{X}_{h|h-1}\right)\right] \tag{14}$$

$$\mathbf{P}_{h|h} = \mathbf{P}_{h|h-1} - \mathbf{GHP}_{h|h-1} \tag{15}$$

end for